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Confirmation and the generalized Nagel–Schaffner model of reduction: a Bayesian analysis

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Abstract In their 2010 (Erkenntnis 73:393–412) paper, Dizadji-Bahmani, Frigg, and Hartmann (henceforth 'DFH') argue that the generalized version of the Nagel–Schaffner model that they have developed (henceforth 'the GNS') is the right one for intertheoretic reduction, i.e. the kind of reduction that involves theories with largely overlapping domains of application. Drawing on the GNS, DFH (Synthese 179:321–338, 2011) presented a Bayesian analysis of the confirmatory relation between the reducing theory and the reduced theory and argued that, post-reduction, evidence confirming the reducing theory also confirms the reduced theory and evidence confirming the reduced theory also confirms the reducing theory, which meets the expectations one has about theories with largely overlapping domains. In this paper, I argue that the Bayesian analysis presented by DFH (Synthese 179:321–338, 2011) faces difficulties. In particular, I argue that the conditional probabilities that DFH introduce to model the bridge law entail consequences that run against the GNS. However, I also argue that, given slight modifications of the analysis that are in agreement with the GNS, one is able to account for these difficulties and, moreover, one is able to more rigorously analyse the confirmatory relation between the reducing and the reduced theory.

Keywords Confirmation · Nagelian reduction · Thermodynamics and statistical mechanics · Bayesian network models

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1 Introduction

Synchronic intertheoretic reduction, that is, "the reductive relation between pairs of theories which have the same (or largely overlapping) domains of application and which are simultaneously valid to various extents," has been an important issue in philosophy of science (DFH 2010, p. 393). A canonical example of purportedly successful reduction of this kind is the reduction of thermodynamics to statistical mechanics (cf. DFH 2010, p. 393, 2011, p. 322; Batterman 2002, pp. 17, 62–63; Sklar 1993, pp. 333ff.). Nagel (1961), and later Schaffner (1967), famously addressed the issue of intertheoretic reduction and offered what is usually called the *Nagelian model of reduction*. The core idea of this model is that a theory T_A reduces a theory T_B if and only if T_B can be logically derived from T_A (or at least that a close cousin of T_B can be logically derived from T_A) with the help of bridge laws. While usually considered as a philosophical background for the purported reduction of statistical mechanics to thermodynamics, this model has been burdened with criticisms that led to a widely shared opinion that the Nagelian model of reduction is untenable and obsolete (e.g. Darden and Maull 1977; Primas 1998; Winther 2009).

However, several defenses of Nagelian reduction have been put forward in recent times. The GNS account is tailored as one. DFH (2010) built on and developed a more sophisticated Nagelian model of reduction (the GNS) that, so the argument goes, successfully accounts for criticisms attached to the Nagelian model of reduction.

One important facet of the GNS is that, given the intertheoretic link that the GNS provides, the two theories are confirmatory of each other: evidence confirming one theory also confirms the other theory. In their 2011 paper DFH analyse this confirmatory relation in terms of Bayesian confirmation theory. In the present paper I argue that this analysis suffers from several difficulties and I propose an alternative Bayesian analysis of the confirmatory relation.

In what follows, I introduce the GNS (Sect. 2.1) and provide an example of reduction to illustrate the workings of the GNS (Sect. 2.2). After that, I discuss philosophical motivations for the GNS (Sect. 2.3), and present a Bayesian analysis of the confirmatory relation between the reducing and the reduced theory given by DFH (2011) (Sect. 2.4). Next, I indicate several difficulties this analysis faces (Sect. 3). I then present an (obvious) revision that accounts for some of the difficulties in the original analysis, but, unfortunately, not all of them (Sect. 4.1). Further, I propose a slightly more modified Bayesian analysis of the confirmatory relation between the theories and argue that, while being more rigorous, it also successfully deals with the problems that DFH's Bayesian analysis faces and it is a better fit to the GNS (Sect. 4.2). Lastly, I present conclusions (Sect. 5).

2 The GNS model and DFH's Bayesian analysis of it

2.1 The generalized Nagel–Schaffner model of reduction

The two theories in the reductive relation are often referred to as the reducing or fundamental theory ($\mathbf{T_F}$) and the reduced or phenomenological theory ($\mathbf{T_P}$). On the

GNS, both $\mathbf{T_F}$ and $\mathbf{T_P}$ have a set of empirical propositions associated with them, namely $\mathcal{T}_F := \{T_F^{(1)}, \ldots, T_F^{(n_F)}\}$ and $\mathcal{T}_P := \{T_P^{(1)}, \ldots, T_P^{(n_P)}\}$, where $(1) \ldots (n_F)$ and $(1) \ldots (n_P)$ are indices (DFH 2010, pp. 397–399, 2011, p. 323). Now, according to this account, the reduction of $\mathbf{T_P}$ to $\mathbf{T_F}$ is captured by the following three steps (DFH 2011, p. 323):

1. Introduce boundary conditions and auxiliary assumption. Using these and $\mathcal{T}_F$ derive a special version of each element $T_F^{(i)}$ in $\mathcal{T}_F$. Dub these $T_F^{*(i)}$ with $\mathcal{T}_F^* := \{T_F^{*(1)}, \ldots, T_F^{*(n_F)}\}$.
2. As $\mathcal{T}_F$ and $\mathcal{T}_P$ are formulated in different vocabularies, in order to connect the terms of the two theories one needs bridge laws. Adopt these laws and substitute terms in $\mathcal{T}_F^*$ according to these laws. This yields a set $\mathcal{T}_P^* := \{T_P^{*(1)}, \ldots, T_P^{*(n_P)}\}$.
3. Show that each element of $\mathcal{T}_P^*$ is strongly analogous to the corresponding element in $\mathcal{T}_P$.[1]

A few remarks on the three steps. First, the boundary conditions and auxiliary assumptions stated in the first step describe the particular setup related to the reducing theory. For instance, in the case of statistical mechanics these are assumptions about mechanical properties of the gas molecules. In order to preclude spurious cases of reduction, DFH (2011, p. 408) impose two caveats on these assumptions: $\mathcal{T}_P$ must not follow from the auxiliary assumptions alone (otherwise the reduction would be trivialized) and auxiliary assumptions cannot be foreign to the conceptual apparatus of $\mathbf{T_F}$ (otherwise the reduction would be cheap as there would be no restrictions on what assumptions are allowable). Second, the status of bridge laws is still highly debated in philosophy of science. On the GNS account they are factual claims posited by scientists working in a particular field (DFH 2010, p. 404, 2011, pp. 328–329).[2] For the purposes of this paper we need concern ourselves with the status of bridge laws. However, it is important to note that as bridge laws, according to the GNS, are posited by different scientists, it could happen that different scientists assign different credences to a particular bridge law (DFH 2011, pp. 328–329). Third, the relationship between $\mathcal{T}_P^*$, on the one side, and $\mathcal{T}_F^*$ and bridge laws, on the other, is of a logical kind: $\mathcal{T}_P^*$ is a deductive consequence of $\mathcal{T}_F^*$ and bridge laws (DFH 2010, pp. 398, 406). Fourth, the notion of strong analogy mentioned in step 3 seems to be a fairly vague one. In order to make it more precise, DFH (2011, p. 409) put the following constrains on $\mathcal{T}_P^*$: it has to share all the essential terms with $\mathcal{T}_P$ and it has to be at least equally empirically adequate as $\mathcal{T}_P$. Lastly, the GNS allows for partial reductions (DFH 2010, p. 399). Namely, if only some terms in $\mathcal{T}_P^*$ are connected to terms in $\mathcal{T}_F^*$ and only some statements of $\mathcal{T}_P^*$ can be deduced from $\mathcal{T}_F^*$ and bridge laws, then the reduction of $\mathbf{T_P}$ still obtains, though a partial one: only those statements that are deduced are reduced.

[1] Note that on the GNS one theory reduces to the other in virtue of empirical propositions (i.e. laws that a theory has). However, proponents of the GNS do not commit themselves to the view that "a theory *just is* [DFH's italics] a set of laws, i.e. $\mathbf{T_A}$ is not identified with $\mathcal{T}_A$" (DFH 2011, p. 323).

[2] An example of the bridge law can be found in Sect. 2.2. Another example of the bridge law is $\mathbf{V} = \mathbf{E}$, where $\mathbf{V}$ is the light vector from the physical optical theory of light and $\mathbf{E}$ is the electric force vector from the theory of electromagnetic radiation. This bridge law is used to derive a number of laws of the physical optical theory of light from Maxwell's equations (see Schaffner 2012, pp. 551–559).

2.2 An example of reduction *à la* GNS

The derivation of the Boyle–Charles Law from the kinetic theory of gases is often mentioned as a clear example of Nagelian-style reduction. In this section I briefly present the derivation and the way it relates to the GNS account of reduction as outlined above.[3]

According to the kinetic theory of gases, a gas is a collection of particles obeying Newton's laws of mechanics. Consider a gas with a large number (n) of sphere-like particles moving in all directions with a fixed mass (m) that interact perfectly elastically with each other and with the walls of a container of volume (V) where the gas is kept. Say we are interested in the force these particles exert on a wall of the container. A way of expressing this force is by talking about the force per unit area; that is to say, we can use the definition of pressure (p) from Newtonian physics: $p = F/A$, where F is the force and A is the area of the wall. Employing the assumption that all particles are perfectly elastic, one is able to show that pressure on the wall is:

$$p = \frac{m\,n}{V}\langle v_x^2\rangle, \tag{1}$$

where v_x is a particle's velocity in x-direction and $\langle v_x^2\rangle$ is the square of v_x averaged over all particles (i.e. $\langle v_x^2\rangle$ is the mean v_x^2 in the gas). Assuming that there is nothing special about x-direction, the average motion of particles in one direction is going to be equal to the average motion of particles in the other two directions:

$$\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle. \tag{2}$$

Since by definition $v^2 = v_x^2 + v_y^2 + v_z^2$, then one can show that:

$$\langle v_x^2\rangle = \frac{1}{3}\langle v_x^2 + v_y^2 + v_z^2\rangle = \frac{1}{3}\langle v^2\rangle. \tag{3}$$

Eq. (1) then becomes:

$$p = \frac{m\,n}{3V}\langle v^2\rangle. \tag{4}$$

From Newtonian mechanics we also have that the kinetic energy of a particle, E_{kin}, is equal to half the square of its velocity times its mass: $E_{\text{kin}} = m\,v^2/2$. Therefore, the mean kinetic energy of a gas is: $\langle E_{\text{kin}}\rangle = m\,\langle v^2\rangle/2$. Substituting in Eq. (4) we finally have:

$$p\,V = \frac{2n}{3}\langle E_{\text{kin}}\rangle. \tag{5}$$

[3] In presenting the derivation, I closely follow Feynman et al. (1964, chapter 39). In parts, I also rely on DFH (2010, pp. 395–396), Dizadji-Bahmani (2011, pp. 31–33, 130–138), Greiner et al. (1997, pp. 6–11), and Pauli (1973, pp. 94–103).

Venturing now into thermodynamics, we find concepts like *temperature* and *(thermal) equilibrium*. We know from experience that if we let two systems with different temperatures interact long enough, they will end up having the same temperature: immersing a hot rod of iron into an ice-cold bucket of water will result in the rod becoming cooler and the water becoming hotter, until eventually both the rod and the water have the same temperature. Equal temperature of two systems (e.g. two gases) then is just the final condition (equilibrium) when they have been interacting with each other long enough.

What can we say about two gases when they are in equilibrium from the point of view of the kinetic theory of gases? To answer this question let us imagine a situation where two gases are in containers separated by a movable frictionless piston. In one container the gas has n_1 particles with mass m_1 and velocity v_1 and in the other container the gas has n_2 particles with mass m_2 and velocity v_2. The bombardment of the piston from one side will result in the piston moving and compressing gas in the other container, which leads to pressure build up in that container, which then leads to more pressure exerted on the piston from that side, which leads to the piston moving and compressing gas in the first container, which leads to pressure build up in that container, and so forth. Eventually, the pressure on the piston from both sides will be equal. Thus, using Eq. (5), the situation in the equilibrium looks as follows:

$$\frac{m_1 n_1}{V_1}\langle v_1^2\rangle = \frac{m_2 n_2}{V_2}\langle v_2^2\rangle \quad \Leftrightarrow \quad \frac{n_1}{V_1}\langle E_{\mathrm{kin}_1}\rangle = \frac{n_2}{V_2}\langle E_{\mathrm{kin}_2}\rangle \tag{6}$$

Can we say something more about the gases in equilibrium than just that the pressures they exert on the piston are equal? The answer is yes. Imagine that the particles in the container on the left developed pressure by having low velocity but high density (i.e. high n/V) and the particles in the container on the right counter that pressure by having high velocity but low density. Though the pressure is the same on both sides, the piston does not stay still: it wiggles since it does not receive a steady pressure. From time to time, the piston will get a big impulse from the right giving more speed to the slower particles on the left. The slower particles will then move faster until they balance the wiggling of the piston (the faster particles on the right will overall lose energy, and consequently speed, to the collisions with the piston). At the equilibrium, the piston is moving at such a mean square speed that it picks up roughly as much energy from the particles as it puts back into them. At that point, the velocities of the two gases will roughly be the same. Hence, at the equilibrium when two gases are at the same temperature, not only are their pressures equal, but their mean kinetic energies are equal as well.[4] This allows us then to define temperature as a function of the mean kinetic energy. However, the scale of temperature has been chosen so that one cannot define temperature simply as the mean kinetic energy without introducing a constant of proportionality. Availing ourselves of one such constant k (Boltzmann's constant), one is able to express temperature in terms of the mean kinetic energy:

[4] Feynman et al. (1964, chapter 39) additionally provide a more comprehensive argument for why the mean kinetic energies of the two gases ought to be equal using only the concepts from the kinetic theory of gases and the definition of equilibrium. For the purposes of this paper, however, we need not go into such detail.

$$T = \frac{2}{3k} \langle E_{\text{kin}} \rangle. \tag{7}$$

Substituting T for $\langle E_{\text{kin}} \rangle$ in Eq. (5) as per Eq. (7), one gets the famous Boyle–Charles Law of thermodynamics:

$$p\,V = n\,k\,T. \tag{8}$$

To summarise the derivation, we started from the kinetic theory of gases and with the help of certain assumptions (e.g. the particles are perfectly elastic and the velocity distribution is isotropic) we showed that Eq. (5) holds. Further, employing the concept of (thermal) equilibrium, we argued that temperature relates to the mean kinetic energy of a gas as in Eq. (7). Ultimately, from Eqs. (5) and (7) we derived the Boyle–Charles Law, given in Eq. (8).

The sketched derivation of the Boyle–Charles Law from the kinetic theory of gases exemplifies the steps that capture the reduction according to the GNS. The reducing theory ($\mathcal{T}_F$) is the kinetic theory of gases and the reduced theory ($\mathcal{T}_P$) is the Boyle–Charles Law. Using $\mathcal{T}_F$ and auxiliary assumptions (e.g. the particles are perfectly elastic and the velocity distribution is isotropic) we derived Eq. (5) ($\mathcal{T}_F^*$), i.e. a special version of $\mathcal{T}_F$ (note that $\mathcal{T}_F^*$ cannot be derived from the auxiliary assumptions alone and that these assumptions are quite natural in the context of $\mathcal{T}_F$). We have, further, argued that one can connect the mean kinetic energy (a term in $\mathcal{T}_F$) and temperature (a term in $\mathcal{T}_P$) via Eq. (7). Equation (7) is then the bridge law. From $\mathcal{T}_F^*$ and the bridge law we derived $\mathcal{T}_P$. In this (simple) case of reduction, $\mathcal{T}_P$ and $\mathcal{T}_P^*$ are one and the same, so we need not show that $\mathcal{T}_P$ and $\mathcal{T}_P^*$ are strongly analogous.

2.3 Why reduce?

In the previous two sections I have outlined one account of reduction and how it applies to a particular case of reduction. But why should scientists be interested in reductions? In the literature one comes across four recurrent reasons for why reduction is desirable: explanation, parsimony, consistency, and confirmation. It is claimed that reductions are a certain kind of explanation (Nagel 1961, p. 338) or, more specifically, that (partial) reductions are causal mechanical explanations (Schaffner 2006, p. 385), where the reducing theory explains the reduced theory; or where the reducing theory explains why the reduced theory seemed correct (Sklar 1967, p. 112); or where the reducing theory explains the phenomena of the reduced theory (van Riel 2014, p. 161); or even where the reducing theory explains the empirical results that the reduced theory fails to explain (Rohrlich 1989, p. 1168). On the GNS, however, explanations, though nice to have, are not the primary aim of reduction; reductions are desirable even if they do not provide explanations (DFH 2010, p. 407).

Parsimony is mentioned in the literature as another desirable product of reduction. Sometimes, reduction can consist in the identification of entities or properties of the reduced theory with entities or properties of the reducing theory, thus simplifying the ontology we adhered to before the reduction (Sklar 1967, pp. 120–121, 1993, pp. 361–362). For instance, a result of the reduction of the physical optical theory of light to the theory of electromagnetic radiation is the identification of light waves with electromagnetic waves: light waves *are* electromagnetic waves (entity identification); or, a result of the reduction of thermodynamics to statistical mechanics is the identification of the temperature of a system with the mean kinetic energy of a system: the temperature *is* the mean kinetic energy (property identification).[5] Now, to establish these identity claims one needs bridge laws that express identity statements. However, on the GNS, bridge laws need not express identity statements: they can express de facto correlations and that would be sufficient for reduction (although not for establishing entity and property identifications). So, on the GNS, parsimony (understood as simplification via entity and property identifications), though perhaps nice to have, is also not the main goal of reduction.

The two main aims of reduction according to the GNS are consistency and confirmation (DFH 2010, pp. 405–406). In the case where two self-consistent and well-confirmed theories with overlapping domains of application provide us with descriptions of the world that contradict each other, one is interested in reconciling these theories so that we end up with a consistent worldview (Nagel 1961, p. 341). Reduction, so the argument goes, can help us to bring together these two theories. To illustrate the point, consider thermodynamics and statistical mechanics. The two theories mostly share the domain of application and are both self-consistent and well-confirmed, but they give descriptions of the world whose mutual consistency we are not sure of. However, reduction makes sure that the two theories become consistent with each other: if $\mathcal{T}_P^*$ (a near enough cousin of $\mathcal{T}_P$, i.e. of thermodynamics in the example) can be deduced from $\mathcal{T}_F^*$ (which is just $\mathcal{T}_F$, i.e. statistical mechanics, plus auxiliary assumptions) and bridge laws, then the two theories are consistent, for deduction is sufficient to establish consistency.

In addition to establishing consistency, reduction also makes sure that, given two theories with largely overlapping domains (like thermodynamics and statistical mechanics), evidence confirming one theory also confirms the other, which is what one would expect to be the case (DFH 2010, p. 406; see also Nagel 1961, p. 361, Sarkar 2015, p. 47, and van Riel 2014, pp. 199–200). The rationale is the following. As on the GNS $\mathcal{T}_P$ and $\mathcal{T}_P^*$ are strongly analogous, supporting evidence for $\mathcal{T}_P$ would also be supporting evidence for $\mathcal{T}_P^*$, and since $\mathcal{T}_P^*$ is a deductive consequence of bridge laws and $\mathcal{T}_F^*$ (i.e. $\mathcal{T}_F$ plus plausible auxiliary assumptions), one would expect that same evidence to confirm $\mathcal{T}_F$. On the other hand, since a deductive consequence of a hypothesis inherits that hypothesis's confirmatory support, evidence supporting $\mathcal{T}_F$ would also support $\mathcal{T}_P^*$, which would in turn support $\mathcal{T}_P$. It is these confirmatory relations between theories that most interest us in the present paper.

[5] It is worth pointing out that in both entity identification and property identification we simplify our previously held ontology not by *eliminating* unnecessary entities or properties of the reduced theory (for instance, eliminating light waves and temperature), but rather by *assimilating* these entities and properties via identification to the corresponding entities and properties of the reducing theory. So, there are still light waves in the world, but instead of two classes of entities—light waves and electromagnetic waves—there is only one (see Sklar 1967, p. 121, 1993, pp. 361–362).

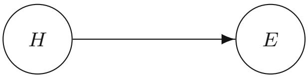

Fig. 1 An example of a Bayesian network

2.4 DFH's Bayesian analysis

In their 2011 paper, DFH argue that the confirmatory relation between the two theories holds if one adopts a Bayesian framework. According to this framework, evidence (E) confirms a hypothesis (H) if $P(\mathrm{H} \mid \mathrm{E}) > P(\mathrm{H})$; E disconfirms H if $P(\mathrm{H} \mid \mathrm{E}) < P(\mathrm{H})$; and E is irrelevant for H if $P(\mathrm{H} \mid \mathrm{E}) = P(\mathrm{H})$. From the probability calculus we further have that if E confirms H, then $l := P(\mathrm{E} \mid \neg\mathrm{H})/P(\mathrm{E} \mid \mathrm{H})$ (known as the *likelihood ratio*) is within the open interval (0, 1); if E disconfirms H, then l is strictly greater than 1; and if E is irrelevant for H, then l is equal to 1. Another common feature of a Bayesian framework is that probabilities do not take extreme values 0 and 1, but lie within the open interval (0, 1). An exception is made, however, in the case of a conditional probability $P(\mathrm{A} \mid \mathrm{B})$ where A is a logical consequence of B; here $P(\mathrm{A} \mid \mathrm{B}) = 1$.[6]

To neatly represent the probabilistic knowledge in a graphical manner, one can employ Bayesian networks.[7] A Bayesian network is a directed acyclic graph (DAG) with nodes representing random variables[8] and arrows representing the relationship between the variables; arrows point only in one direction (hence directed graph) and there is no path that starts at a certain node and, following the arrows, ends at the same one (hence acyclic graph). For instance, the network in Fig. 1 is a Bayesian network: it has two nodes representing two random variables H and E each taking two values: H and ¬H *and* E and ¬E, respectively. The one arrow going from H to E encodes the probabilistic relationship between the two variables: E is probabilistically dependent on H (the reason could be that, for instance, E is more likely to obtain if H is the case then if ¬H is the case). To specify the network, one needs to set the prior probabilities to all root nodes, i.e. nodes that do not have incoming arrows, and one needs to set the conditional probabilities of all other nodes, given their respective nodes at the other end of the incoming arrows. In the network in Fig. 1 we need to set the prior probability of the root node H, i.e. we need to fix $P(\mathrm{H})$, and the conditional probabilities of the node E given the node H, i.e. we need to fix $P(\mathrm{E} \mid \mathrm{H})$ and $P(\mathrm{E} \mid \neg\mathrm{H})$.

As an illustration let us consider the following example. Say H is the proposition 'S has cancer' and E is the proposition 'The test is positive'. The Bayesian network in Fig. 1 would then represent the probabilistic relation between S having/not having cancer and the test being positive/negative. $P(\mathrm{H})$ would be the physician's prior degree of belief—i.e. her degree of belief before seeing the test results—that the patient S has cancer (it could be, for instance, just a proportion of people in the population

[6] For surveys on Bayesianism see Háyek and Hartmann (2010) and Hartmann and Sprenger (2011). For a critical discussion of Bayesianism see Earman (1992).

[7] For an introduction to Bayesian networks see Pearl (1988), Neapolitan (2003), Bovens and Hartmann (2003, pp. 67ff.), DFH (2011, p. 325).

[8] Throughout the article, random variables in the network are binary; that is, some random variable A (denoted by italicized letters) can take two values A or ¬A (denoted by non-italicized letters).

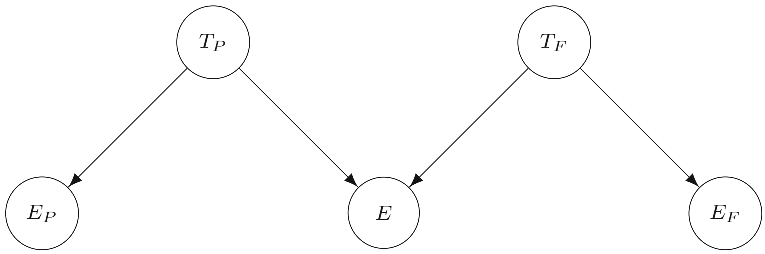

Fig. 2 The Bayesian network representing the situation before the reduction

that have cancer). $P(\mathrm{E} \mid \mathrm{H})$ would be the true positive rate (the rate of people with cancer that the test correctly identified as such) and $P(\mathrm{E} \mid \neg\mathrm{H})$ would be the false positive rate (the rate of healthy people that the test incorrectly identified as having cancer). Using Bayes' Theorem[9], one could then calculate $P(\mathrm{H} \mid \mathrm{E})$. If we learn that the test is positive and if $P(\mathrm{H} \mid \mathrm{E}) > P(\mathrm{H})$, then, by Bayesian confirmation theory, the hypothesis that the patient has cancer is confirmed by the test being positive.

Using the formal machinery of Bayesian networks, DFH model situations before and after the reduction. For simplicity, the authors assume that both $\mathcal{T}_F$ and $\mathcal{T}_P$ contain only one element, namely $\mathrm{T_F}$ and $\mathrm{T_P}$ respectively. Also, in addition to evidence ($\mathrm{E_F}$) that confirms $\mathrm{T_F}$ and evidence ($\mathrm{E_P}$) that confirms $\mathrm{T_P}$, the authors include in the Bayesian network evidence (E) that confirms both $\mathrm{T_F}$ and $\mathrm{T_P}$. This is justified by the existence of real world examples of such evidence (DFH 2011, p. 324). Putting all this together, the situation before the reduction is depicted in Fig. 2.

The relevant probabilities that specify this network are:

$$\begin{aligned} P_1(\mathrm{T_F}) = t_F,&\quad P_1(\mathrm{T_P}) = t_P \\ P_1(\mathrm{E_F} \mid \mathrm{T_F}) = p_F,&\quad P_1(\mathrm{E_F} \mid \neg\mathrm{T_F}) = q_F \\ P_1(\mathrm{E_P} \mid \mathrm{T_P}) = p_P,&\quad P_1(\mathrm{E_P} \mid \neg\mathrm{T_P}) = q_P \\ P_1(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P}) = \alpha,&\quad P_1(\mathrm{E} \mid \mathrm{T_F}, \neg\mathrm{T_P}) = \beta \\ P_1(\mathrm{E} \mid \neg\mathrm{T_F}, \mathrm{T_P}) = \gamma,&\quad P_1(\mathrm{E} \mid \neg\mathrm{T_F}, \neg\mathrm{T_P}) = \delta \end{aligned} \tag{9}$$

After the reduction, the situation is different: two more nodes (T_F^* and T_P^*) are added to the network. This is represented in Fig. 3.

The relevant probabilities for this network include all from Eq. (9), with the exception of $P_1(\mathrm{T_P}) = t_P$. As T_P is no longer a root node, instead of $P_1(\mathrm{T_P}) = t_P$ we now have:

$$P_2(\mathrm{T_P} \mid \mathrm{T_P^*}) = p_P^*, \quad P_2(\mathrm{T_P} \mid \neg\mathrm{T_P^*}) = q_P^* \tag{10}$$

In addition to these probabilities, to complete the network DFH (2011, p. 328) also specify the following probabilities:

9 Bayes' Theorem: $P(\mathrm{H} \mid \mathrm{E}) = \frac{P(\mathrm{E}|\mathrm{H})P(\mathrm{H})}{P(\mathrm{E})} = \frac{P(\mathrm{E}|\mathrm{H})P(\mathrm{H})}{P(\mathrm{E}|\mathrm{H})P(\mathrm{H})+P(\mathrm{E}|\neg\mathrm{H})P(\neg\mathrm{H})}$.

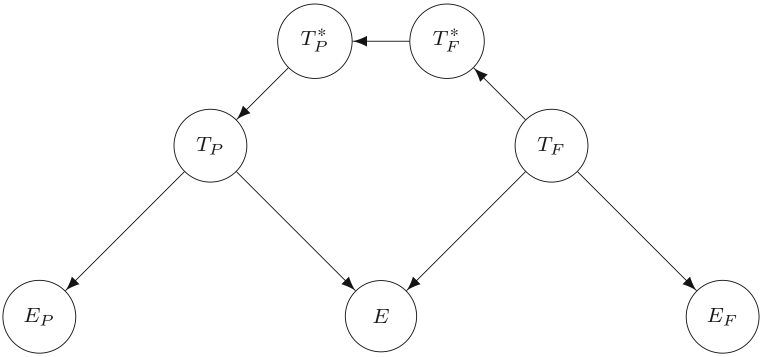

Fig. 3 The Bayesian network representing the situation after the reduction

$$P_2(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{F}) = p_F^*, \quad P_2(\mathrm{T}_\mathrm{F}^* \mid \neg\mathrm{T}_\mathrm{F}) = q_F^* \tag{11}$$

$$P_2(\mathrm{T}_\mathrm{P}^* \mid \mathrm{T}_\mathrm{F}^*) = 1, \quad P_2(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*) = 0 \tag{12}$$

The conditional probabilities in Eq. (12) assume extreme values since they represent the bridge law: T_P^* is a logical consequence of T_F^*, supposing the bridge law in the background.

Given the Bayesian network in Fig. 3 and the probability assignments related to that network, one can prove that after the reduction the following two theorems hold (DFH 2011, p. 329):

Theorem 1 *E_F confirms T_P iff* $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Theorem 2 *E_P confirms T_F iff* $(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

The two theorems entail that E_F confirms T_P and E_P confirms T_F if E_F confirms T_F (in which case the likelihood ratio q_F/p_F is within the interval (0, 1) and, therefore, $p_F > q_F$), E_P confirms T_P ($p_P > q_P$), T_F confirms T_F^* ($p_F^* > q_F^*$), and T_P^* confirms T_P ($p_P^* > q_P^*$). That E_F confirms T_F and E_P confirms T_P has been assumed from the beginning. That T_F confirms T_F^* and T_P^* confirms T_P seems to be plausible according to DFH (2011, p. 329) as the confirmation flow from T_F to T_P is thereby ensured. So, given the Bayesian network in Fig. 3, the related probabilities, and the assumptions about confirmatory relations among T_F, T_P, T_F^*, T_P^*, E_F, and E_P, one can show that post-reduction E_F confirms T_P and E_P confirms T_F.

3 Critical discussion

In this section I point to some difficulties faced by the Bayesian analysis presented in the previous section.

DFH (2011, p. 327) correctly point out that the following conditional independencies hold in the network representing the situation *before* the reduction (Fig. 2):[10]

$$E_F \perp\!\!\!\perp T_P \mid T_F, \quad E_P \perp\!\!\!\perp T_F \mid T_P \tag{13}$$

Now, DFH (2011, p. 328) also claim that the conditional independencies in Eq. (13) do not hold in the Bayesian network representing the situation *after* the reduction (Fig. 3). However, the conditional independencies in Eq. (13) do hold after the reduction. Looking at the Bayesian network in Fig. 3 we observe that there are two possible paths between E_F and T_P: $E_F - T_F - T_F^* - T_P^* - T_P$ and $E_F - T_F - E - T_P$. As both paths are blocked at T_F by $\{T_F\}$, then E_F and T_P are d-separated[11] by $\{T_F\}$; hence, $E_F \perp\!\!\!\perp T_P \mid T_F$ holds. By similar reasoning we get that $E_P \perp\!\!\!\perp T_F \mid T_P$ also holds.

Further, the authors (2011, p. 327) also claim that the following equalities are a direct consequence of the conditional independencies in Eq. (13):

$$P_1(\mathrm{T_P} \mid \mathrm{E_F}) = P_1(\mathrm{T_P}), \quad P_1(\mathrm{T_F} \mid \mathrm{E_P}) = P_1(\mathrm{T_F}) \tag{14}$$

However, the equations in (14) do not follow directly from the independencies in Eq. (13). A counter-example is actually the Bayesian network that represents the situation *after* the reduction (Fig. 3). Here, as shown in the previous paragraph, the independencies from Eq. (13) also hold, but the equations from (14) (which can also be translated into independencies, namely unconditional independencies $\mathrm{T_P} \perp\!\!\!\perp \mathrm{E_F}$ and $\mathrm{T_F} \perp\!\!\!\perp \mathrm{E_P}$) do not, since now neither T_P and E_F nor T_F and E_P are unconditionally independent. In spite of that, the equations in (14) do hold before the reduction: they follow from the independencies in Eq. (13) *coupled with* $T_F \perp\!\!\!\perp T_P$ that also holds in the Bayesian network in Fig. 2; alternatively, the equations in (14) can be derived more directly using d-separation (see "Appendix" for more details).

The difficulties that I have pointed at so far do not undercut the main project laid out by DFH, as one can still show that after the reduction $\mathrm{E_F}$ confirms $\mathrm{T_P}$ and $\mathrm{E_P}$ confirms $\mathrm{T_F}$. However, the authors also claim that random variables T_F^* and T_P^* are "*qua* the bridge law, intersubstitutable with each other" and that the arrow "could have also been drawn from T_P^* to T_F^*," in which case "we had to require $P(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 1$ and $P(\mathrm{T_F^*} \mid \neg\mathrm{T_P^*}) = 0$" (DFH 2011, pp. 329–330). Let us, then, modify the Bayesian network in Fig. 3 by now drawing an arrow from T_P^* to T_F^* instead of an arrow that goes from T_F^* to T_P^* (Fig. 4).

Analyzing the network in Fig. 4, we first note that T_P^* is now a root node and as such it has a prior probability that one needs to specify. However, it seems at best odd that we now have to assign a prior probability $P(\mathrm{T_P^*})$, since $\mathrm{T_P^*}$ it is tightly related to $\mathrm{T_F^*}$ and the bridge law, of which it is a logical consequence, and to $\mathrm{T_P}$ via strong analogy.

[10] '$A \perp\!\!\!\perp B \mid C$' encodes the information that A and B are conditionally independent given C. By definition, A and B are conditionally independent given C, i.e. $A \perp\!\!\!\perp B \mid C$, if and only if $P(\mathrm{A} \mid \mathrm{B}, \mathrm{C}) = P(\mathrm{A} \mid \mathrm{C})$.

[11] d-separation is a property of Bayesian networks by which one can track down all the independences (conditional and unconditional ones) in the Bayesian network: $A \perp\!\!\!\perp B \mid C$ if and only if A and B are d-separated by $\{C\}$. Two nodes A and B are d-separated by $\{C\}$ if all the paths in the network between A and B are blocked by $\{C\}$. For more details on d-separation see Neapolitan (2003, pp. 70ff.).

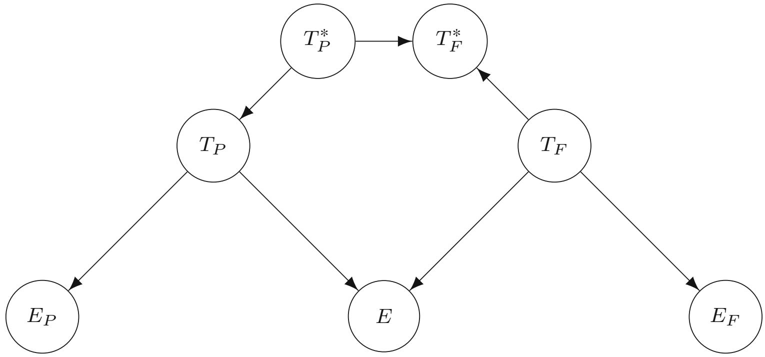

Fig. 4 The Bayesian network representing the situation after the reduction with an *arrow* from T_P^* to T_F^*

Second, it is incorrect that we only need to specify $P(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{P}^*)$ and $P(\mathrm{T}_\mathrm{F}^* \mid \neg\mathrm{T}_\mathrm{P}^*)$ in the case of an arrow flip between T_F^* and T_P^*: T_F^* is now probabilistically dependent on both T_P^* and T_F, so we would need to specify $P(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{F}, \mathrm{T}_\mathrm{P}^*)$, $P(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{F}, \neg\mathrm{T}_\mathrm{P}^*)$, $P(\mathrm{T}_\mathrm{F}^* \mid \neg\mathrm{T}_\mathrm{F}, \mathrm{T}_\mathrm{P}^*)$, and $P(\mathrm{T}_\mathrm{F}^* \mid \neg\mathrm{T}_\mathrm{F}, \neg\mathrm{T}_\mathrm{P}^*)$. However, it is not clear what values these probabilities should assume. Should any have as a value 1 or 0? How do we model the bride law in this case? Third and most importantly, paths $E_F - T_F - T_F^* - T_P^* - T_P$ and $E_F - T_F - E - T_P$ are now both blocked by $\varnothing$ at T_F^* and E respectively, since there are converging arrows both at T_F^* and E. Hence, E_F and T_P are d-separated by $\varnothing$ and $E_F \perp\!\!\!\perp T_P$ holds. Similarly, we get that $E_P \perp\!\!\!\perp T_F$ holds. This implies that $P(\mathrm{T}_\mathrm{P} \mid \mathrm{E}_\mathrm{F}) = P(\mathrm{T}_\mathrm{P})$ and $P(\mathrm{T}_\mathrm{F} \mid \mathrm{E}_\mathrm{P}) = P(\mathrm{T}_\mathrm{F})$. So, *after* the reduction we have the same pair of equations from (14) that describe the situation *before* the reduction and say that E_F does not confirm (or disconfirm) T_P and that E_P does not confirm (or disconfirm) T_F. But this runs against the GNS account which aims at establishing these confirmatory relations *after* the reduction.

One could say to this that, surely, if one flips the arrow, undesirable results emerge. But the GNS talks of T_P^* being a logical consequence of T_F^* (supposing the bridge law) and not the other way around. This, then, gives a reason to fix the direction of the arrow. Another reason to fix the arrow direction is that on the GNS, to deduce T_P^* from T_F plus auxiliary assumptions (and thus establish the consistency of T_P^* and T_F) all we need is that whenever T_F applies, then T_P^* applies as well (the other direction, i.e. whenever T_P^* applies, then T_F applies as well, is not necessary for deduction). So, DFH could then simply add a note to their Bayesian analysis saying that, given the two aforementioned reasons, the network in Fig. 3 (i.e. the network from their actual analysis), but not the network in Fig. 4 (i.e. the network that they suggest would also do the job), is the way to model the situation after the reduction. Granting this point, the analysis suffers from at least three further problems.

Problem 1. From the discussion in Sect. 2.2 of the reduction of the Boyle–Charles Law to the kinetic theory of gases we learned that in this (simple) case of reduction T_P^* and T_P are one and the same and encode Eq. (8), namely $p\,V = n\,k\,T$. From the equations in Eq. (12), i.e. ones that model the bridge law, we have that $P_2(\mathrm{T}_\mathrm{P} \mid \mathrm{T}_\mathrm{F}^*) = 1$, where, in this particular case, T_F^* represents Eq. (5), namely $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle$. Since,

$P_2(\mathrm{T_P} \mid \mathrm{T_F^*}) = 1$, then $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle \vDash p\,V = n\,k\,T$; but this is clearly false. A better way of writing the entailment would be: $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle \vDash_B \; p\,V = n\,k\,T$, where B is Eq. (7), namely $T = \frac{2}{3k}\langle E_{\mathrm{kin}}\rangle$. In other words, supposing B, $\mathrm{T_F^*}$ entails $\mathrm{T_P}$. But this seems to suggest that the probability distribution P_2 needs to be modified so as to incorporate B in the background, i.e. B needs to be a part of the probability function P_2, in order for the entailment to hold. So, instead of $P_2(\mathrm{T_P} \mid \mathrm{T_F^*}) = 1$ we should write $P_B(\mathrm{T_P} \mid \mathrm{T_F^*}) = 1$ to denote that B is part of the probability function. This means that besides $P_B(\mathrm{T_P} \mid \mathrm{T_F^*}) = 1$ and $P_B(\mathrm{T_P} \mid \neg\mathrm{T_F^*}) = 0$, we also have to specify $P_B(\mathrm{T_F})$, $P_B(\mathrm{E_F} \mid \mathrm{T_F})$, $P_B(\mathrm{E_P} \mid \mathrm{T_P})$, $P_B(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P})$, etc. Intuitively, the unconditional probability of $\mathrm{T_F}$ and the conditional probabilities $\mathrm{E_F}$ given $\mathrm{T_F}$, $\mathrm{E_P}$ given $\mathrm{T_P}$, and E given $\mathrm{T_F}$ and $\mathrm{T_P}$ should be the same before and after the reduction: the fact that we have reduced $\mathrm{T_P}$ to $\mathrm{T_F}$ should not affect the unconditional probability of the reducing theory or the conditional probabilities of the three kinds of evidence. DFH also seem to share this intuition since on their account $P_1(\mathrm{T_F}) = P_2(\mathrm{T_F})$, $P_1(\mathrm{E_F} \mid \mathrm{T_F}) = P_2(\mathrm{E_F} \mid \mathrm{T_F})$, $P_1(\mathrm{E_P} \mid \mathrm{T_P}) = P_2(\mathrm{E_P} \mid \mathrm{T_P})$, and $P_1(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P}) = P_2(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P})$ hold. However, if, as I argued, after the reduction we should have the probability function P_B instead of P_2, then $P_1(\mathrm{E_P} \mid \mathrm{T_P}) = P_B(\mathrm{E_P} \mid \mathrm{T_P})$ and $P_1(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P}) = P_B(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P})$ are no longer guaranteed to hold. For, as on the GNS we derived $\mathrm{T_P}$ from $\mathrm{T_F^*}$ and B, then after the reduction $\mathrm{T_P}$ is dependent on B. This implies that, contrary to our intuitions, $P_B(\mathrm{E_P} \mid \mathrm{T_P})$ and $P_B(\mathrm{E} \mid \mathrm{T_F}, \mathrm{T_P})$ may change after the reduction.

Problem 2. The reduction relation should be asymmetric: though the Boyle–Charles Law reduces to the kinetic theory of gases, the kinetic theory of gases does *not* reduce to the Boyle–Charles Law.[12] From Eq. (12) it follows that $P_2(\mathrm{T_P^*} \mid \mathrm{T_F^*}) = P_2(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 1$ and $P_2(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}) = P_2(\mathrm{T_F^*} \mid \neg\mathrm{T_P^*}) = 0$ (see "Appendix"). Hence, not only does $\mathrm{T_F^*}$ entail $\mathrm{T_P^*}$, but also $\mathrm{T_P^*}$ entails $\mathrm{T_F^*}$ (supposing the bridge law). So, (i) the proposed Bayesian analysis requires symmetry in reduction: in our example, not only do we have a reduction of the Boyle–Charles Law, but Eq. (5) also reduces to the Boyle–Charles Law (i.e. Eq. (8)). This seems to run against the intuition that the reduction should go only in one direction. Further, (ii) the mutual entailment of $\mathrm{T_P^*}$ and $\mathrm{T_F^*}$ would prevent partial reductions—cases of reduction where not all the laws of a theory are reduced, but only some laws of a theory, namely those that are deduced, are reduced—which the GNS allows for: in the general case, the mutual entailment implies that *all* laws of the $\mathcal{T}_P^*$ can be deduced from $\mathcal{T}_F^*$, given the bridge laws. The Bayesian analysis that DFH put forward accounts, thus, only for the the cases of complete reduction; yet, not only does the GNS allow for the partial reductions, but some authors have argued that it is partial reductions that we can best hope for in sciences like biology (Schaffner 2006, p. 384). This means that the proposed Bayesian analysis is not the best fit for the GNS and that it may disregard certain sciences when it comes to the subject of reduction.

Problem 3. From Eq. (12) it also follows that $P_2(\mathrm{T_F^*}) = P_2(\mathrm{T_P^*})$ (see "Appendix"). As on the GNS $\mathrm{T_P^*}$ is a deductive consequence of $\mathrm{T_F^*}$ *and* the bridge laws, it seems rather unlikely, though possible, that $P_2(\mathrm{T_F^*}) = P_2(\mathrm{T_P^*})$. To illustrate the point, let us once

[12] A number of authors support the claim that the reduction relation should be asymmetric: Kuipers (1982), Sarkar (2015, p. 47), Riel (2013), Riel and Gulick (2016, p. 18).

again return to our example of reduction. Since T_P^* and T_P are the same, the equality then translates as $P_2(\mathrm{T}_\mathrm{F}^*) = P_2(\mathrm{T}_\mathrm{P})$. In other words, after the reduction the probability of $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle$ has to be equal to the probability of $p\,V = n\,k\,T$. It seems somewhat implausible that the values of the two probabilities always be the same, since, for one, the two equations hold under different assumptions: $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle$ is a deductive consequence of the kinetic theory of gases coupled with the auxiliary assumptions and $p\,V = n\,k\,T$ is a deductive consequence of $p\,V = \frac{2n}{3}\langle E_{\mathrm{kin}}\rangle$ and the bridge laws. This then suggests that the relation between $P_2(\mathrm{T}_\mathrm{F}^*)$ and $P_2(\mathrm{T}_\mathrm{P}^*)$ should best be left open: $P_2(\mathrm{T}_\mathrm{F}^*)$ could be greater than, less than, or equal to $P_2(\mathrm{T}_\mathrm{P}^*)$.

These three problems should not be perceived as knockdown arguments against DFH's Bayesian analysis: after all, we are in the model-building business where knock-down arguments arguably do not have as much bite. The problems' main purpose is rather to invite and motivate an alternative model of the situation after the reduction. My main goal is thus to present another way of modeling the situation after the reduction that helps us address the three problems, that better fits the GNS, and that allows us to rigorously show some new results regarding the confirmatory relations that we could only presume to hold with the original analysis, or so I argue.

4 A revised Bayesian analysis

In this section I try to account for the difficulties presented in the previous section by introducing modifications to DFH's Bayesian analysis. I start by discussing a natural revision to the original analysis and, after recognizing that it does not account for all the problems of the original analysis, I present an alternative Bayesian analysis.

4.1 An (unsuccessful) easy remedy

Carefully examining the difficulties, one notes that most of them are due to the probability assignments in Eq. (12). So, as an amendment one could consider changing these assignments. $P_3(\mathrm{T}_\mathrm{P}^* \mid \mathrm{T}_\mathrm{F}^*)$ has to be equal to 1 as T_P^* is a logical consequence of T_F^* (supposing the bridge law in the background). But what about $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*)$? Saying that T_F^* entails T_P^* probabilistically demands only that $P_3(\mathrm{T}_\mathrm{P}^* \mid \mathrm{T}_\mathrm{F}^*) = 1$. It does not put constrains on the value of $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*)$. However, letting $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*)$ take the value of 1 would run against the motivation of the GNS, since on this account T_P^* is a logical consequence of T_F^* and the bridge law, and allowing $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*)$ to assume the value of 1 would mean that T_P^* is also a logical consequence of $\neg\mathrm{T}_\mathrm{F}^*$ and the bridge law. On the other hand, allowing $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*)$ to take the value 0 would bring us back to the problems from the previous section. So, one can specify $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*) = a$, where $a \in (0, 1)$.

With $P_3(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*) = a$ instead of $P_2(\mathrm{T}_\mathrm{P}^* \mid \neg\mathrm{T}_\mathrm{F}^*) = 0$ and everything else as in the original analysis, one can prove the following theorems:

Theorem 3 *E_F confirms T_P iff* $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Theorem 4 *E_P confirms T_F iff* $(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Notice that these theorems are of exactly the same form as Theorems 1 and 2 from the original analysis. Moreover, one can show that $0 < P_3(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}) < 1$, i.e. T^*_P does not entail T^*_F. This successfully answers *Problem 2* of the original analysis. However, *Problem 1* is still present since $P_3(\mathrm{T}^*_\mathrm{P} \mid \mathrm{T}^*_\mathrm{F}) = 1$ holds and, therefore, one can run the same argument as in the previous section. In addition, one can also show that $P_3(\mathrm{T}^*_\mathrm{P}) > P_3(\mathrm{T}^*_\mathrm{F})$, which is an unfortunate result in light of *Problem 3*, as the relation between $P_3(\mathrm{T}^*_\mathrm{P})$ and $P_3(\mathrm{T}^*_\mathrm{F})$ is again held fixed (see the "Appendix" for more details on this subsection).

4.2 An alternative Bayesian analysis

The previous attempt to account for the problems, however, seems to be on the right track. So, we need to look for additional modifications. Naturally, we can further investigate bridge laws. In the original analysis, the bridge law is supposed in the background without explicitly being included in the network. On the other hand, DFH (2011, pp. 328–329) also mention that (a) different scientists can have different credences about a particular bridge law. This claim seems to be vindicated by the following two observations. First, in the derivation of the Boyle–Charles Law from Sect. 2.2 following Feynman et al. (1964) I presented an argument for the claim that temperature can be expressed in terms of mean kinetic energy. Some scientists may be more and some may be less convinced by this argument and so their credences may vary with respect to the bridge law in Eq. (7). Second, bridge laws can be empirically tested. For instance, Ager et al. (1974) cite Joule's experiments which can be used to vindicate the relation between temperature and mean kinetic energy. Schaffner (2012) cites Hertz's and Wiener's experiments that helped establish the bridge law $\mathbf{V} = \mathbf{E}$ used in the derivation of many laws of the physical optical theory of light from the theory of electromagnetic radiation. Since one scientist's confidence in the empirical support of the bridge law may be different from another scientist's confidence, the credences scientists have about a particular bridge law may differ from one scientist to another and may change through time. Furthermore, DFH (2011, p. 329) conjecture that (b) the flow of confirmation in the network is dependent on the probability value one assigns to the bridge law: with lower probability value the degree of confirmation of $\mathrm{T_F}$ by $\mathrm{E_P}$ or $\mathrm{T_P}$ by $\mathrm{E_F}$ is lower. Together, (a) and (b) then give reason to explicitly model, i.e. to endogenously define, the bridge law (B) in the network (cf. Bovens and Hartmann 2003, pp. 56ff.). What is more, given (a) it seems plausible that scientists can give prior probabilities to a particular bridge law; this in turn allows us to model B as a root node in the network. The question, now, is how to connect B with other nodes. Well, since T^*_P is a logical consequence of T^*_F and B, it is natural to draw arrows from T^*_F and from B to T^*_P. Putting it all together, the new Bayesian network representing the state of affairs after the reduction is depicted in Fig. 5.

Since there is now an additional node in the network, besides probabilities specified for the Bayesian network in Fig. 3, we also need to specify $P_4(\mathrm{B})$. So,

$$P_4(\mathrm{B}) = b \tag{15}$$

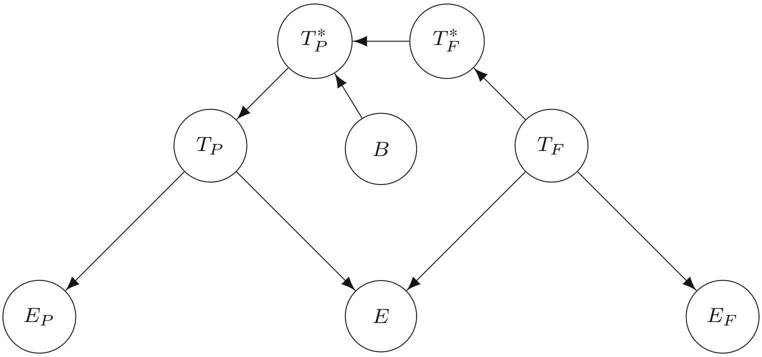

Fig. 5 The Bayesian network representing the situation after the reduction with the bridge law defined endogenously

Further, instead of the probabilities in Eq. (12), we now have to assign values to $P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \mathrm{B})$, $P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \mathrm{B})$, $P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \neg\mathrm{B})$, and $P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \neg\mathrm{B})$. As $\mathrm{T_P^*}$ is a logical consequence of $\mathrm{T_F^*}$ and B, then $P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \mathrm{B}) = 1$. But what about $P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \mathrm{B})$, $P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \neg\mathrm{B})$, and $P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \neg\mathrm{B})$? Drawing on the rationale from Sect. 4.1, we do not assign them value 1 as we do not want to say that $\mathrm{T_P^*}$ is also entailed by $\neg\mathrm{T_F^*}$ or $\neg$B. We do not assign them value 0 either, since *Problem 2* of the original analysis reemerges (see "Appendix"). So, we assign them value a, where $a \in (0, 1)$.[13] Therefore, instead of Eq. (12) we now have:

$$\begin{gathered} P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \mathrm{B}) = 1 \\ P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \mathrm{B}) = P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \neg\mathrm{B}) = P_4(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \neg\mathrm{B}) = a \end{gathered} \tag{16}$$

Given the new probability assignments, one is able to show that $0 < P_4(\mathrm{T_F^*} \mid \mathrm{T_P^*}, \mathrm{B}) < 1$, i.e. $\mathrm{T_F^*}$ is not entailed by $\mathrm{T_P^*}$ and B, which successfully addresses *Problem 2* of the original analysis where $\mathrm{T_P^*}$ entails $\mathrm{T_F^*}$ (supposing the bridge law). But we had that result in Sect. 4.1 as well. What about the relation between $P_4(\mathrm{T_F^*})$ and $P_4(\mathrm{T_P^*})$, that is, *Problem 3*? One finds that on the new probability assignments it is left open, i.e. $P_4(\mathrm{T_P^*})$ can be greater than, less than, or equal to $P_4(\mathrm{T_F^*})$, depending on the particular values one assigns to the relevant probabilities (see "Appendix"). So, *Problem 3* is successfully addressed as well. Further, *Problem 1* does not emerge since we have that $P_4(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \mathrm{B}) = 1$ and $\mathrm{T_F^*}, \mathrm{B} \vDash \mathrm{T_P^*}$ is true without us having to suppose something additionally in order for the entailment to hold. Hence, the values of the conditional probabilities $\mathrm{E_P}$ given $\mathrm{T_P}$ and E given $\mathrm{T_P}$ and $\mathrm{T_F}$ remain the same before and after the reduction. Therefore, all three problems that one can ascribe to DFH's Bayesian analysis do not emerge in the revised Bayesian analysis. What is more, as I have made use of the real-world example of reduction (i.e. the reduction of the Boyle–Charles Law to the kinetic theory of gases from Sect. 2.2) to motivate

[13] Although a, in principle, can take any value in the open interval (0, 1), it seems more plausible that it assumes a rather low value since we do not expect to often find that $\mathrm{T_P^*}$ holds and that $\neg\mathrm{T_F^*}$ or $\neg$B hold.

and inform the three problems, this alternative Bayesian analysis is arguably a better fit for the real-world examples of reduction than DFH's analysis.

Furthermore, given the Bayesian network in Fig. 5 and probability assignments associated with it, one can prove the following theorems:

Theorem 5 *E_F confirms T_P iff* $(p_F - q_F)\,(p^*_F - q^*_F)\,(p^*_P - q^*_P) > 0$.

Theorem 6 *E_P confirms T_F iff* $(p_P - q_P)\,(p^*_F - q^*_F)\,(p^*_P - q^*_P) > 0$.

As in Sect. 4.1, these theorems are in exactly the same form as Theorem 1 and Theorem 2, the main results from the original analysis, which say that E_F confirms T_P and E_P confirms T_F. Recently, Sarkar (2015, p. 47) pointed out that though this result is a good start, one may also be interested in whether there is any added confirmation: after the reduction, does E_F add anything to E_P's confirmation of T_P and does E_P add anything to E_F's confirmation of T_F? In relation to this question one can prove the following theorems:

Theorem 7 *E_F adds to E_P's confirmation of T_P iff* $(p_F - q_F)\,(p^*_F - q^*_F)\,(p^*_P - q^*_P) > 0$.

Theorem 8 *E_P adds to E_F's confirmation of T_F iff* $(p_P - q_P)\,(p^*_F - q^*_F)\,(p^*_P - q^*_P) > 0$.

The two theorems entail that, after the reduction, E_F enhances the confirmation of T_P and E_P enhances the confirmation of T_F under the same conditions under which E_F confirms T_P and E_P confirms T_F.[14] Thus, not only does E_F confirm T_P and E_P confirm T_F, but also E_F provides additional confirmational boost to T_P (that is, E_F provides the confirmational boost to T_P that is in addition to that of E_P) and, similarly, E_P provides additional confirmational boost to T_F (that is, E_P provides the confirmational boost to T_F that is in addition to that of E_F).

The analysis so far implies that whether E_F confirms T_P or E_P confirms T_F and whether E_F enhances the confirmation of T_P or E_P enhances the confirmation of T_F does not depend on the value that one assigns to $P_4(\mathrm{B})$. However, if we are interested in the degree of confirmation[15] of T_P by E_F or in the degree of confirmation of T_F by E_P, then one can prove the following two theorems:

Theorem 9 *Given a, p_F, q_F, p^*_F, q^*_F, p^*_P, q^*_P, and t_F are constant and $p_F > q_F$, $p^*_F > q^*_F$, and $p^*_P > q^*_P$, if b increases (decreases), then $d(T_P, E_F)$ increases (decreases).*

Theorem 10 *Given a, p_P, q_P, p^*_F, q^*_F, p^*_P, q^*_P, and t_F are constant and $p_P > q_P$, $p^*_F > q^*_F$, and $p^*_P > q^*_P$, if b increases (decreases), then $d(T_F, E_P)$ increases (decreases).*

The two theorems say that, other values remaining the same, by increasing (decreasing) the value of $P_4(\mathrm{B})$, the degree of confirmation of T_P and the degree of confirmation

14 Interestingly, but perhaps unsurprisingly, one can show that in DFH's original analysis the two theorems hold in exactly the same form (see Theorem 7' and Theorem 8' in "Appendix").

15 Here I use the *difference* measure d as the measure of degree of confirmation of a hypothesis (H) by evidence (E): $d(\mathrm{H}, \mathrm{E}) := P(\mathrm{H} \mid \mathrm{E}) - P(\mathrm{H})$ (cf. Fitelson 1999, p. 363; Hartmann and Sprenger, 2011, p. 613).

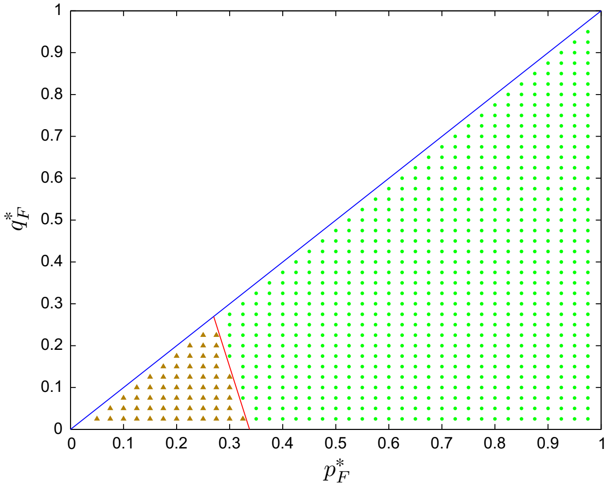

Fig. 6 Dependence of $P_4(\mathrm{T_P^*}) - P_4(\mathrm{T_F^*})$ on p_F^* and q_F^*, with $a = 0.1$, $b = 0.7$, $t_F = 0.8$, and $p_F^* > q_F^*$. As $p_F^* > q_F^*$, a point (p_F^*, q_F^*) can only be inside the area below the *blue diagonal* (*dotted green bullet* and *sandal triangle*). When a point (p_F^*, q_F^*) lies on the *red line*, then $P_4(\mathrm{T_P^*}) - P_4(\mathrm{T_F^*}) = 0$, i.e. $P_4(\mathrm{T_P^*}) = P_4(\mathrm{T_F^*})$. The area *left* of the *red line* (*dotted sandal triangle*) corresponds to $P_4(\mathrm{T_P^*}) > P_4(\mathrm{T_F^*})$. The area *right* of the *red line* (*dotted green bullet*) corresponds to $P_4(\mathrm{T_P^*}) < P_4(\mathrm{T_F^*})$. (Color figure online)

of $\mathrm{T_F}$ increases (decreases). Or in other words, the degree of confirmation of $\mathrm{T_P}$ and the degree of confirmation of $\mathrm{T_F}$ are directly proportional to the value of $P_4(\mathrm{B})$, given that other values are constant. This is an additional improvement to both the original analysis and the easy remedy from Sect. 4.1 where one could only speculate on the relation between the confidence we have in the bridge law and the degree of confirmation of $\mathrm{T_P}$ and $\mathrm{T_F}$ (see DFH 2011, p. 329). Thus, the new alternative Bayesian analysis is richer in content than both the easy remedy and DFH's analysis.

Next, although the present Bayesian analysis allows $P_4(\mathrm{T_P^*})$ to be greater than, less than, or equal to $P_4(\mathrm{T_F^*})$, given the following plausible value assignments:

$$a = 0.1, b = 0.7, t_F = 0.8, \text{ and } p_F^* > q_F^*,$$

as a further result one finds that $P_4(\mathrm{T_P^*}) < P_4(\mathrm{T_F^*})$ is more likely than $P_4(\mathrm{T_P^*}) \geq P_4(\mathrm{T_F^*})$. Specifically, $P_4(\mathrm{T_F^*})$ is always greater than $P_4(\mathrm{T_P^*})$ if the difference between p_F^* and q_F^* is sufficiently high (around 0.3) or if both p_F^* and q_F^* assume values greater than approx. 0.35, as shown in Fig. 6.[16]

This result is very much in agreement with the GNS. Since $\mathrm{T_F^*}$ is a deductive consequence of $\mathrm{T_F}$ and plausible auxiliary assumptions which are not foreign to $\mathrm{T_F}$, one would expect $P_4(\mathrm{T_F^*})$ to be close to $P_4(\mathrm{T_F})$; that is, one would expect $P_4(\mathrm{T_F^*})$ to assume a relatively high value (otherwise, if $P_4(\mathrm{T_F})$ would not be sufficiently high, we would not be in the business of reducing $\mathrm{T_P}$ to $\mathrm{T_F}$). Further, as $\mathrm{T_P^*}$ is a deductive consequence of $\mathrm{T_F^*}$ and plausible bridge laws (B), one would expect $P_4(\mathrm{T_P^*})$ to be

[16] The situation does not differ much given somewhat different value assignments for a, b, and t_F.

relatively close to $P_4(\mathrm{T}_\mathrm{F}^*)$ but not higher than $P_4(\mathrm{T}_\mathrm{F}^*)$ since (1) $P_4(\mathrm{T}_\mathrm{F}^*)$ takes a high value and (2) T_P^* is not a direct consequence of only T_F^* but of T_F^* and B. This, however, is not to say that we would expect in all cases to find out that $P_4(\mathrm{T}_\mathrm{P}^*)$ is less than $P_4(\mathrm{T}_\mathrm{F}^*)$, but rather that, given different parameters (e.g. different value assignments for $P_4(\mathrm{T}_\mathrm{F})$, $P_4(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{F})$, and so on), what we expect to find is that in the majority of cases $P_4(\mathrm{T}_\mathrm{P}^*)$ turns out to be less than $P_4(\mathrm{T}_\mathrm{F}^*)$.

In addition to the theorems so far mentioned, one can also prove the following theorem that is in exactly the same form as the Theorem 3 from DFH's analysis (DFH 2011, p. 331):

Theorem 11 $\Delta_0 = 0$ *iff* $(p_F^* = q_F^*)$ *or* $(p_P^* = q_P^*)$*. And* $\Delta_0 > 0$ *if* $(p_F^* > q_F^*)$ *and if* $(p_P^* > q_P^*)$*.*

In this theorem, where $\Delta_0 := P_4(\mathrm{T_F}, \mathrm{T_P}) - P_1(\mathrm{T_F}, \mathrm{T_P})$, the conjunction of $\mathrm{T_F}$ and $\mathrm{T_P}$ is compared before and after the reduction and it is said that if either $\mathrm{T_F}$ and T_F^* or T_P^* and $\mathrm{T_P}$ are independent, then $\mathrm{T_F}$ and $\mathrm{T_P}$ remain independent after the reduction; and if $\mathrm{T_F}$ confirms T_F^* and if T_P^* confirms $\mathrm{T_P}$, then the conjunction of $\mathrm{T_F}$ and $\mathrm{T_P}$ is more likely after the reduction.

Now, DFH (2011) prove other important theorems that describe the relationship of the posterior probabilities of the conjunction of $\mathrm{T_F}$ and $\mathrm{T_P}$ and the relationship of the prior and posterior probabilities of the conjunction of $\mathrm{T_F}$ and $\mathrm{T_P}$, which are not proven in this paper. However, it seems plausible to conjecture that these theorems can be derived given the Bayesian network in Fig. 5 and probability assignments associated to it as well. For instance, Theorems 4 and 6 of DFH's analysis say, loosely put, that if either $\mathrm{T_F}$ and T_F^* or T_P^* and $\mathrm{T_P}$ are independent, then there is no flow of confirmation between $\mathrm{T_F}$ and $\mathrm{T_P}$ and the situation after the reduction is the same as the one before the reduction (DFH 2011, pp. 331–332). Since the difference between the network in Fig. 5 and the network in Fig. 3 is in the introduction of *B* in the network in Fig. 5 and, as mentioned above, *B* by itself does not stop the flow of confirmation between $\mathrm{T_F}$ and $\mathrm{T_P}$, then one would expect the other theorems to also hold in this revised analysis.

5 Conclusion

In this paper, I have argued that a Bayesian analysis of the confirmatory relation between $\mathrm{T_P}$ and $\mathrm{E_F}$ and between $\mathrm{T_F}$ and $\mathrm{E_P}$ presented by DFH (2011) is not without difficulties. I have shown that the arrow flip between T_P^* and T_F^* would render the situation after the reduction exactly like the one before the reduction. Moreover, I have argued that the probability function that DFH use to model the situation after the reduction should be modified so that it incorporates the bridge law. However, this leads to the undesirable consequence that the conditional probabilities $\mathrm{E_P}$ given $\mathrm{T_P}$ and E given $\mathrm{E_F}$ and $\mathrm{E_P}$ may not remain the same after the reduction (*Problem 1*). Furthermore, it follows from DFH's analysis that T_P^* and T_F^* entail each other. This mutual entailment (i) requires symmetry in reduction (contrary to one's expectation that reduction should be asymmetric) and (ii) it prevents partial reductions (which the GNS explicitly allows for and which may be the only kind of reduction that we find in sciences like biology) (*Problem 2*). From DFH's analysis it also follows that

$P_2(\mathrm{T_P^*}) = P_2(\mathrm{T_F^*})$ holds. This is, however, unlikely to always be the case and the relation between $P_2(\mathrm{T_P^*})$ and $P_2(\mathrm{T_F^*})$ should then best be left open (*Problem 3*).

As a remedy, one could specify $P_3(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}) = a$, where $a \in (0, 1)$. Although with this revised version, dubbed an easy remedy, one can prove Theorems 3 and 4 that are in exactly the same form as the Theorems 1 and 2 form the original analysis and one can account for *Problem 2*, alas, *Problems 1* and *3* remain unanswered.

I then introduced an alternative Bayesian analysis where further modification is made: a node modeling the bridge laws is added to the Bayesian network. I have argued that given this new Bayesian network and related probability assignments, one is able to successfully address the drawbacks of the original analysis that I have pointed out (i.e. *Problems 1*, *2*, and *3*), thus making the new Bayesian analysis more realistic than DFH's analysis since the three problems were motivated and informed by the real-world example of reduction. Also, one can prove Theorems 5, 6 and 11 that share the form with the first three theorems of the original analysis. Furthermore, in contrast to both the original analysis and the easy remedy, one can now explicitly show (Theorems 9 and 10) the relation between the flow of confirmation and the value one assigns to the probability of the bridge law, making this analysis richer in content. Also, I have shown that though the new Bayesian analysis allows $P_2(\mathrm{T_P^*})$ to be greater than, less than, or equal to $P_2(\mathrm{T_F^*})$, it is not completely silent about the relation between $P_2(\mathrm{T_P^*})$ and $P_2(\mathrm{T_F^*})$: one finds that, given plausible value assignments, $P_4(\mathrm{T_P^*}) < P_4(\mathrm{T_F^*})$ is more likely than $P_4(\mathrm{T_P^*}) \geq P_4(\mathrm{T_F^*})$, which, as I have argued, is in agreement with the GNS. In addition, I have shown not only that $\mathrm{E_F}$ confirms $\mathrm{T_P}$ and $\mathrm{E_P}$ confirms $\mathrm{T_F}$ after the reduction, but also that $\mathrm{E_F}$ enhances the confirmation of $\mathrm{T_P}$ (Theorem 7) and $\mathrm{E_P}$ enhances the confirmation of $\mathrm{T_F}$ (Theorem 8). Lastly, I have conjectured that one should expect to prove the other theorems from the original analysis that describe the important confirmatory relations after the reduction.

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Appendix

To show: $E_F \perp\!\!\!\perp T_P \mid T_F$ and $T_P \perp\!\!\!\perp T_F$ entail $P_1(\mathrm{T_P} \mid \mathrm{E_F}) = P_1(\mathrm{T_P})$[17]

$$E_F \perp\!\!\!\perp T_P \mid T_F \tag{1}$$

$$T_P \perp\!\!\!\perp E_F \mid T_F \quad \text{(by the s-g axiom Symmetry)} \tag{2}$$

$$P_1(\mathrm{T_P} \mid \mathrm{E_F}, \mathrm{T_F}) = P_1(\mathrm{T_P} \mid \mathrm{T_F}) \quad \text{(by the def. of cond. independence)} \tag{3}$$

$$T_P \perp\!\!\!\perp T_F \tag{4}$$

$$P_1(\mathrm{T_P} \mid \mathrm{T_F}) = P_1(\mathrm{T_P}) \quad \text{(by the def. of independence)} \tag{5}$$

[17] The expression 's-g axiom' stands for semi-graphoid axiom. For more details on semi-graphoid axioms see Pearl (1988, pp. 84ff).

$$P_1(\mathrm{T_P} \mid \mathrm{E_F}, \mathrm{T_F}) = P_1(\mathrm{T_P}) \quad \text{(from (3) and (5))} \tag{6}$$
$$T_P \perp\!\!\!\perp E_F, T_F \quad \text{(by the def. of cond. independence)} \tag{7}$$
$$T_P \perp\!\!\!\perp E_F \quad \text{(by the s-g axiom Decomposition)} \tag{8}$$
$$P_1(\mathrm{T_P} \mid \mathrm{E_F}) = P_1(\mathrm{T_P}) \quad \text{(by the def. of independence)} \tag{9}$$

□

Similarly, we get that $E_P \perp\!\!\!\perp T_F \mid T_P$ and $T_P \perp\!\!\!\perp T_F$ entail $P_1(\mathrm{T_F} \mid \mathrm{E_P}) = P_1(\mathrm{T_F})$.

To show: $P_1(\mathrm{T_P} \mid \mathrm{E_F}) = P_1(\mathrm{T_P})$—*by d-separation*

There is only one possible path between E_F and T_P, namely $E_F - T_F - E - T_P$, which is blocked at E by $\varnothing$. Therefore, $T_P \perp\!\!\!\perp E_F$. By the definition of independence this translates into $P_1(\mathrm{T_P} \mid \mathrm{E_F}) = P_1(\mathrm{T_P})$. □

Similarly, we get that $P_1(\mathrm{T_F} \mid \mathrm{E_P}) = P_1(\mathrm{T_F})$ holds by d-separation before the reduction.

To show: $P_2(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 1$

$$P_2(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}) = 0$$
$$P_2(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}) = \frac{P_2(\mathrm{T_P^*}, \neg\mathrm{T_F^*})}{P_2(\neg\mathrm{T_F^*})} = 0$$
$$P_2(\mathrm{T_P^*}, \neg\mathrm{T_F^*}) = 0$$
$$P_2(\neg\mathrm{T_F^*} \mid \mathrm{T_P^*}) = \frac{P_2(\mathrm{T_P^*}, \neg\mathrm{T_F^*})}{P_2(\mathrm{T_P^*})} = 0$$
$$P_2(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 1$$

□

To show: $P_2(\mathrm{T_F^*} \mid \neg\mathrm{T_P^*}) = 0$

$$P_2(\mathrm{T_P^*} \mid \mathrm{T_F^*}) = 1$$
$$P_2(\neg\mathrm{T_P^*} \mid \mathrm{T_F^*}) = \frac{P_2(\neg\mathrm{T_P^*}, \mathrm{T_F^*})}{P_2(\mathrm{T_F^*})} = 0$$
$$P_2(\neg\mathrm{T_P^*}, \mathrm{T_F^*}) = 0$$
$$P_2(\mathrm{T_F^*} \mid \neg\mathrm{T_P^*}) = \frac{P_2(\neg\mathrm{T_P^*}, \mathrm{T_F^*})}{P_2(\neg\mathrm{T_P^*})} = 0$$

□

To show: $P_2(\mathrm{T_F^*}) = P_2(\mathrm{T_P^*})$

$$P_2(\mathrm{T_P^*} \mid \mathrm{T_F^*}) = \frac{P_2(\mathrm{T_P^*}, \mathrm{T_F^*})}{P_2(\mathrm{T_F^*})} = 1$$
$$P_2(\mathrm{T_P^*}, \mathrm{T_F^*}) = P_2(\mathrm{T_F^*})$$

$$P_2(\mathrm{T}_\mathrm{F}^* \mid \mathrm{T}_\mathrm{P}^*) = \frac{P_2(\mathrm{T}_\mathrm{P}^*, \mathrm{T}_\mathrm{F}^*)}{P_2(\mathrm{T}_\mathrm{P}^*)} = \frac{P_2(\mathrm{T}_\mathrm{F}^*)}{P_2(\mathrm{T}_\mathrm{P}^*)} = 1$$
$$P_2(\mathrm{T}_\mathrm{F}^*) = P_2(\mathrm{T}_\mathrm{P}^*)$$

□

I adopt the following convention: $\overline{z} := 1 - z$.

Theorem 3 E_F confirms T_P iff $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$P_3(\mathrm{T_P}|\mathrm{E_F}) = \frac{P_3(\mathrm{T_P}, \mathrm{E_F})}{P_3(\mathrm{E_F})}$$
$$P_3(\mathrm{T_P}, \mathrm{E_F}) = \sum_{T_P^*, T_F^*, T_F} P_3(\mathrm{T_P} \mid T_P^*) P_3(T_P^* \mid T_F^*) P_3(T_F^* \mid T_F) P_3(T_F) \cdot P_3(\mathrm{E_F} \mid T_F)$$
$$= p_F\, t_F\, (p_P^*\, p_F^* + a\, p_P^*\, \overline{p_F^*} + \overline{a}\, q_P^*\, \overline{p_F^*}) + q_F\, \overline{t_F}\, (p_P^*\, q_F^* + a\, p_P^*\, \overline{q_F^*} + \overline{a}\, q_P^*\, \overline{q_F^*})$$
$$P_3(\mathrm{E_F}) = \sum_{T_F} P_3(\mathrm{E_F} \mid T_F) P_3(T_F) = p_F\, t_F + q_F\, \overline{t_F}$$
$$P_3(\mathrm{T_P}) = \sum_{T_P^*, T_F^*, T_F} P_3(\mathrm{T_P} \mid T_P^*) P_3(T_P^* \mid T_F^*) P_3(T_F^* \mid T_F) P_3(T_F)$$
$$= t_F\, (p_P^*\, p_F^* + a\, p_P^*\, \overline{p_F^*} + \overline{a}\, q_P^*\, \overline{p_F^*}) + \overline{t_F}\, (p_P^*\, q_F^* + a\, p_P^*\, \overline{q_F^*} + \overline{a}\, q_P^*\, \overline{q_F^*})$$
$$P_3(\mathrm{T_P} \mid \mathrm{E_F}) - P_3(\mathrm{T_P}) = \frac{\overline{a}\, t_F\, \overline{t_F}\, (p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{p_F\, t_F + q_F\, \overline{t_F}}$$

□

Theorem 4 E_P confirms T_F iff $(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$P_3(\mathrm{T_F} \mid \mathrm{E_P}) = \frac{P_3(\mathrm{T_F}, \mathrm{E_P})}{P_3(\mathrm{E_P})}$$
$$P_3(\mathrm{T_F}, \mathrm{E_P}) = P_3(\mathrm{T_F}) \sum_{T_P^*, T_F^*, T_P} P_3(\mathrm{E_P} \mid T_P) P_3(T_P \mid T_P^*) P_3(T_P^* \mid T_F^*) \cdot P_3(T_F^* \mid \mathrm{T_F})$$
$$= t_F \left[(p_F^* + a\, \overline{p_F^*})\,(p_P\, p_P^* + q_P\, \overline{p_P^*}) + \overline{a}\, \overline{p_F^*}\, (p_P\, q_P^* + q_P\, \overline{q_P^*}) \right]$$
$$P_3(\mathrm{E_P}) = \sum_{T_P^*, T_F^*, T_P, T_F} P_3(\mathrm{E_P} \mid T_P) P_3(T_P \mid T_P^*) P_3(T_P^* \mid T_F^*) P_3(T_F^* \mid T_F)$$

$$
\begin{aligned}
&\cdot P_3(T_F) \\
&= \Big(t_F\,(p_F^* + a\,\overline{p_F^*}) + \overline{t_F}\,(q_F^* + a\,\overline{q_F^*})\Big)\,(p_P\,p_P^* + q_P\,\overline{p_P^*}) \\
&\quad + \overline{a}\,(\overline{p_F^*}\,t_F + \overline{q_F^*}\,\overline{t_F})\,(p_P\,q_P^* + q_P\,\overline{q_P^*})
\end{aligned}
$$

$$P_3(\mathrm{T_F}) = t_F$$

$$P_3(\mathrm{T_F} \mid \mathrm{E_P}) - P_3(\mathrm{T_F}) = \frac{\overline{a}\,t_F\,\overline{t_F}\,(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{P_3(\mathrm{E_P})}$$

□

To show: $0 < P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) < 1$

$$
\begin{aligned}
P_3(\mathrm{T_P^*} \mid \mathrm{T_F^*}) &= \frac{P_3(\mathrm{T_P^*}, \mathrm{T_F^*})}{P_3(\mathrm{T_F^*})} = 1 \\
P_3(\mathrm{T_P^*}, \mathrm{T_F^*}) &= P_3(\mathrm{T_F^*}) \\
P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) &= \frac{P_3(\mathrm{T_P^*}, \mathrm{T_F^*})}{P_3(\mathrm{T_{P)}^*}} = \frac{P_3(\mathrm{T_F^*})}{P_3(\mathrm{T_P^*})} \\
P_3(\mathrm{T_P^*}) &= \sum_{T_F^*} P_3(\mathrm{T_P^*} \mid T_F^*) P_3(T_F^*) \\
&= P_3(\mathrm{T_F^*}) + a\,P_3(\neg\mathrm{T_F^*}) = P_3(\mathrm{T_F^*})\,\overline{a} + a \\
P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) &= \frac{P_3(\mathrm{T_F^*})}{P_3(\mathrm{T_F^*})\,\overline{a} + a}
\end{aligned}
$$

Suppose $P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 0$, then

$$
\begin{aligned}
\frac{P_3(\mathrm{T_F^*})}{P_3(\mathrm{T_F^*})\,\overline{a} + a} &= 0 \\
P_3(\mathrm{T_F^*}) &= 0
\end{aligned}
$$

But $P_3(\mathrm{T_F^*})$ cannot be equal to 0, since by assumption all probabilities are within the open interval (0, 1) (except for the conditional ones that encode logical consequence).

Suppose $P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) = 1$, then

$$
\begin{aligned}
\frac{P_3(\mathrm{T_F^*})}{P_3(\mathrm{T_F^*})\,\overline{a} + a} &= 1 \\
P_3(\mathrm{T_F^*}) &= P_3(\mathrm{T_F^*})\,\overline{a} + a \\
P_3(\mathrm{T_F^*}) - P_3(\mathrm{T_F^*})\,\overline{a} &= a \\
P_3(\mathrm{T_F^*})\,a &= a \\
P_3(\mathrm{T_F^*}) &= 1
\end{aligned}
$$

But $P_3(\mathrm{T_F^*})$ cannot be equal to 1, for the reason mentioned above.

Hence,

$$0 < P_3(\mathrm{T_F^*} \mid \mathrm{T_P^*}) < 1$$

□

To show: $P_3(\mathrm{T_P^*}) > P_3(\mathrm{T_F^*})$

$$\begin{aligned}
P_3(\mathrm{T_P^*}) &= \sum_{T_F^*} P_3(\mathrm{T_P^*} \mid T_F^*) P_3(T_F^*) \\
&= P_3(\mathrm{T_F^*}) + a\, P_3(\neg\mathrm{T_F^*}) \\
P_3(\mathrm{T_P^*}) - P_3(\mathrm{T_F^*}) &= P_3(\mathrm{T_F^*}) + a\, P_3(\neg\mathrm{T_F^*}) - P_3(\mathrm{T_F^*}) \\
&= a\, P_3(\neg\mathrm{T_F^*}) > 0
\end{aligned}$$

□

Of $P_5(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \mathrm{B})$, $P_5(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \neg\mathrm{B})$, and $P_5(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \neg\mathrm{B})$, as the most plausible candidate for assigning the value 0 is $P_5(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \mathrm{B})$, since, one could say, given true bridge laws and false T_F^* (i.e. $\neg\mathrm{T_F^*}$), theory T_P^* should not come out as true. As for $P_5(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \neg\mathrm{B})$ and $P_5(\mathrm{T_P^*} \mid \neg\mathrm{T_F^*}, \neg\mathrm{B})$, regard them as randomizers and assign them $a \in (0, 1)$ (see Bovens and Hartmann 2003, pp. 57ff.). However, with these probability assignments, a drawback of the original analysis recurs: $\mathrm{T_P^*}$ and B entail $\mathrm{T_F^*}$. In order to show that this entailment holds, observe that in the network in Fig. 5 $T_F^* \perp\!\!\!\perp B$ holds (the only two paths between T_F^* and B, i.e. $T_F^* - T_P^* - B$ and $T_F^* - T_F - E - T_P - T_P^* - B$, are blocked by $\varnothing$ at T_P^* and E respectively; so, T_F^* and B are d-separated by $\varnothing$).

To show: $P_5(\mathrm{T_F^*} \mid \mathrm{T_P^*}, \mathrm{B}) = 1$

$$P_5(\mathrm{T_P^*} \mid \mathrm{T_F^*}, \mathrm{B}) = \frac{P_5(\mathrm{T_P^*}, \mathrm{T_F^*}, \mathrm{B})}{P_5(\mathrm{T_F^*}, \mathrm{B})} = 1 \tag{10}$$

$$P_5(\mathrm{T_P^*}, \mathrm{T_F^*}, \mathrm{B}) = P_5(\mathrm{T_F^*}, \mathrm{B}) \tag{11}$$

$$T_F^* \perp\!\!\!\perp B \tag{12}$$

$$P_5(\mathrm{T_F^*}, \mathrm{B}) = P_5(\mathrm{T_F^*}) P_5(\mathrm{B}) \tag{13}$$

$$P_5(\mathrm{T_P^*}, \mathrm{T_F^*}, \mathrm{B}) = P_5(\mathrm{T_F^*}) P_5(\mathrm{B}) \qquad \text{(from (11) and (13))} \tag{14}$$

$$P_5(\mathrm{T_F^*} \mid \mathrm{T_P^*}, \mathrm{B}) = \frac{P_5(\mathrm{T_P^*}, \mathrm{T_F^*}, \mathrm{B})}{P_5(\mathrm{T_P^*}, \mathrm{B})} = \frac{P_5(\mathrm{T_F^*}) P_5(\mathrm{B})}{P_5(\mathrm{T_P^*}, \mathrm{B})} \tag{15}$$

$$P_5(\mathrm{T_P^*}, \mathrm{B}) = P_5(\mathrm{B}) \sum_{T_F^*, T_F} P_5(\mathrm{T_P^*} \mid T_F^*, \mathrm{B}) P_5(T_F^* \mid T_F) P_5(T_F) \tag{16}$$

$$= b\,(p_F^*\, t_F + q_F^*\, \overline{t_F}) \tag{17}$$

$$P_5(\mathrm{T_F^*}) = \sum_{T_F} P_5(\mathrm{T_F^*} \mid T_F) P_5(T_F) \tag{18}$$

$$= p_F^*\, t_F + q_F^*\, \overline{t_F} \tag{19}$$

$$P_5(\mathrm{T}^*_\mathrm{P}, \mathrm{B}) = b\, P_5(\mathrm{T}^*_\mathrm{F}) \qquad \text{(from (17) and (19))} \tag{20}$$

$$P_5(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}, \mathrm{B}) = \frac{b\, P_5(\mathrm{T}^*_\mathrm{F})}{b\, P_5(\mathrm{T}^*_\mathrm{F})} = 1 \qquad \text{(from (15) and (20))} \tag{21}$$

□

To show: $0 < P_4(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}, \mathrm{B}) < 1$

$$P_4(\mathrm{T}^*_\mathrm{P} \mid \mathrm{T}^*_\mathrm{F}, \mathrm{B}) = \frac{P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{T}^*_\mathrm{F}, \mathrm{B})}{P_4(\mathrm{T}^*_\mathrm{F}, \mathrm{B})} = 1 \tag{22}$$

$$P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{T}^*_\mathrm{F}, \mathrm{B}) = P_4(\mathrm{T}^*_\mathrm{F}, \mathrm{B}) \tag{23}$$

$$T^*_F \perp\!\!\!\perp B \tag{24}$$

$$P_4(\mathrm{T}^*_\mathrm{F}, \mathrm{B}) = P_4(\mathrm{T}^*_\mathrm{F}) P_4(\mathrm{B}) \tag{25}$$

$$P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{T}^*_\mathrm{F}, \mathrm{B}) = P_4(\mathrm{T}^*_\mathrm{F}) P_4(\mathrm{B}) \qquad \text{(from (23) and (25))} \tag{26}$$

$$P_4(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}, \mathrm{B}) = \frac{P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{T}^*_\mathrm{F}, \mathrm{B})}{P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{B})} = \frac{P_4(\mathrm{T}^*_\mathrm{F}) P_4(\mathrm{B})}{P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{B})} \tag{27}$$

$$P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{B}) = P_4(\mathrm{B}) \sum_{T^*_F, T_F} P_4(\mathrm{T}^*_\mathrm{P} \mid T^*_F, \mathrm{B}) P_4(T^*_F \mid T_F) P_4(T_F) \tag{28}$$

$$= b\, \left((p^*_F\, t_F + q^*_F\, \overline{t_F})\, \overline{a} + a\right) \tag{29}$$

$$P_4(\mathrm{T}^*_\mathrm{F}) = \sum_{T_F} P_4(\mathrm{T}^*_\mathrm{F} \mid T_F) P_4(T_F) \tag{30}$$

$$= p^*_F\, t_F + q^*_F\, \overline{t_F} \tag{31}$$

$$P_4(\mathrm{T}^*_\mathrm{P}, \mathrm{B}) = b\left(P_4(\mathrm{T}^*_\mathrm{F})\, \overline{a} + a\right) \qquad \text{(from (29) and (31))} \tag{32}$$

$$P_4(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}, \mathrm{B}) = \frac{b\, P_4(\mathrm{T}^*_\mathrm{F})}{b\left(P_4(\mathrm{T}^*_\mathrm{F})\, \overline{a} + a\right)} \qquad \text{(from (27) and (32))} \tag{33}$$

$$= \frac{P_4(\mathrm{T}^*_\mathrm{F})}{P_4(\mathrm{T}^*_\mathrm{F})\, \overline{a} + a} \tag{34}$$

$$0 < \frac{P_4(\mathrm{T}^*_\mathrm{F})}{P_4(\mathrm{T}^*_\mathrm{F})\, \overline{a} + a} < 1 \qquad \text{(from the proof of } 0 < P_3(\mathrm{T}^*_\mathrm{F} \mid \mathrm{T}^*_\mathrm{P}) < 1) \tag{35}$$

□

To show: $P_4(\mathrm{T}^*_\mathrm{P}) > P_4(\mathrm{T}^*_\mathrm{F})$ or $P_4(\mathrm{T}^*_\mathrm{P}) < P_4(\mathrm{T}^*_\mathrm{F})$ or $P_4(\mathrm{T}^*_\mathrm{P}) = P_4(\mathrm{T}^*_\mathrm{F})$

$$P_4(\mathrm{T}^*_\mathrm{P}) = \sum_{T^*_F, B, T_F} P_4(\mathrm{T}^*_\mathrm{P} \mid T^*_F, B) P_4(B) P_4(T^*_F \mid T_F) P_4(T_F)$$

$$= t_F\, \left(b\, p^*_F + a\, \overline{b}\, p^*_F + a\, \overline{p^*_F}\right) + \overline{t_F}\, \left(b\, q^*_F + a\, \overline{b}\, q^*_F + a\, \overline{q^*_F}\right)$$

$$P_4(\mathrm{T}^*_\mathrm{F}) = \sum_{T_F} P_4(\mathrm{T}^*_\mathrm{F} \mid T_F) P_4(T_F)$$

$$= p^*_F\, t_F + q^*_F\, \overline{t_F}$$

$$\begin{aligned} P_4(\mathrm{T_P^*}) - P_4(\mathrm{T_F^*}) &= a - (\overline{a}\,\overline{b} + a)\,(p_F^*\,t_F + q_F^*\,\overline{t_F}) \\ &= a - (\overline{a}\,\overline{b} + a)\,P_4(\mathrm{T_F^*}) \end{aligned}$$

So, when $P_4(\mathrm{T_F^*}) = \frac{a}{\overline{a}\,\overline{b}+a}$, then $P_4(\mathrm{T_P^*}) = P_4(\mathrm{T_F^*})$ (note that for $0 < a < 1$ and $0 < b < 1$, $0 < \frac{a}{\overline{a}\,\overline{b}+a} < 1$). When $P_4(\mathrm{T_F^*}) < \frac{a}{\overline{a}\,\overline{b}+a}$, then $P_4(\mathrm{T_P^*}) > P_4(\mathrm{T_F^*})$. When $P_4(\mathrm{T_F^*}) > \frac{a}{\overline{a}\,\overline{b}+a}$, then $P_4(\mathrm{T_P^*}) < P_4(\mathrm{T_F^*})$.

□

Theorem 5 E_F confirms T_P iff $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$\begin{aligned} P_4(\mathrm{T_P} \mid \mathrm{E_F}) &= \frac{P_4(\mathrm{T_P}, \mathrm{E_F})}{P_4(\mathrm{E_F})} \\ P_4(\mathrm{T_P}, \mathrm{E_F}) &= \sum_{T_P^*, T_F^*, B, T_F} P_4(\mathrm{T_P} \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B) P_4(T_F^* \mid T_F) P_4(T_F) \\ &\quad \cdot P_4(\mathrm{E_F} \mid T_F) \\ &= p_F\,t_F\,\left(p_P^*\,(b\,p_F^* + a\,\overline{b} + a\,b\,\overline{p_F^*}) + q_P^*\,(\overline{a}\,\overline{b} + \overline{a}\,b\,\overline{p_F^*})\right) \\ &\quad + q_F\,\overline{t_F}\,\left(p_P^*\,(b\,q_F^* + a\,\overline{b} + a\,b\,\overline{q_F^*}) + q_P^*\,(\overline{a}\,\overline{b} + \overline{a}\,b\,\overline{q_F^*})\right) \\ P_4(\mathrm{E_F}) &= \sum_{T_F} P_4(\mathrm{E_F} \mid T_F) P_4(T_F) \\ &= p_F\,t_F + q_F\,\overline{t_F} \\ P_4(\mathrm{T_P}) &= \sum_{T_P^*, T_F^*, B, T_F} P_4(\mathrm{T_P} \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B) P_4(T_F^* \mid T_F) P(T_F) \\ &= p_P^*\,\left(b\,p_F^*\,t_F + b\,q_F^*\,\overline{t_F} + a\,\overline{b} + a\,b\,\overline{p_F^*}\,t_F + a\,b\,\overline{q_F^*}\,\overline{t_F}\right) \\ &\quad + q_P^*\,\left(\overline{a}\,\overline{b} + \overline{a}\,b\,\overline{p_F^*}\,t_F + \overline{a}\,b\,\overline{q_F^*}\,\overline{t_F}\right) \\ P_4(\mathrm{T_P} \mid \mathrm{E_F}) - P_4(\mathrm{T_P}) &= \frac{\overline{a}\,b\,t_F\,\overline{t_F}\,(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{p_F\,t_F + q_F\,\overline{t_F}} \end{aligned}$$

□

Theorem 6 E_P confirms T_F iff $(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$\begin{aligned} P_4(\mathrm{T_F} \mid \mathrm{E_P}) &= \frac{P_4(\mathrm{T_F}, \mathrm{E_P})}{P_4(\mathrm{E_P})} \\ P_4(\mathrm{T_F}, \mathrm{E_P}) &= P_4(\mathrm{T_F}) \sum_{T_P^*, T_F^*, B, T_P} P_4(\mathrm{E_P} \mid T_P) P_3(T_P \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B) \\ &\quad \cdot P_4(T_F^* \mid \mathrm{T_F}) \end{aligned}$$

$$= t_F \left[(b\, p_F^* + a\, \overline{b} + a\, b\, \overline{p_F^*})\, (p_P\, p_P^* + q_P\, \overline{p_P^*}) \right.$$
$$\left. + \overline{a}\, (\overline{b} + b\, \overline{p_F^*})\, (p_P\, q_P^* + q_P\, \overline{q_P^*}) \right]$$

$$P_4(\mathrm{E_P}) = \sum_{T_P^*, T_F^*, B, T_P, T_F} P_4(\mathrm{E_P} \mid T_P) P_4(T_P \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B)$$
$$\cdot\, P_4(T_F^* \mid T_F) P_4(T_F)$$
$$= (b\, p_F^*\, t_F + b\, q_F^*\, \overline{t_F} + a\, \overline{b} + a\, b\, \overline{p_F^*}\, t_F + a\, b\, \overline{q_F^*}\, \overline{t_F})\, (p_P\, p_P^* + q_P\, \overline{p_P^*})$$
$$+\ \overline{a}\, (\overline{b} + b\, \overline{p_F^*}\, t_F + b\, \overline{q_F^*}\, \overline{t_F})\, (p_P\, q_P^* + q_P\, \overline{q_P^*})$$

$$P_4(\mathrm{T_F}) = t_F$$

$$P_4(\mathrm{T_F} \mid \mathrm{E_P}) - P_4(\mathrm{T_F}) = \frac{\overline{a}\, b\, t_F\, \overline{t_F}\, (p_P - q_P)\, (p_F^* - q_F^*)\, (p_P^* - q_P^*)}{P_4(\mathrm{E_P})}$$

□

Theorem 7 E_{F} adds to E_{P}'s confirmation of T_{P} iff $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$P_4(\mathrm{T_P} \mid \mathrm{E_P}, \mathrm{E_F}) = \frac{P_4(\mathrm{T_P}, \mathrm{E_P}, \mathrm{E_F})}{P_4(\mathrm{E_P}, \mathrm{E_F})}$$

$$P_4(\mathrm{T_P}, \mathrm{E_P}, \mathrm{E_F}) = P_4(\mathrm{E_P} \mid \mathrm{T_P}) \sum_{T_P^*, T_F^*, B, T_F} P_4(\mathrm{T_P} \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B)$$
$$\cdot\, P_4(T_F^* \mid T_F) P_4(\mathrm{E_F} \mid T_F) P_4(T_F)$$
$$= p_P \left[\overline{a}\, b\, (p_F^*\, p_F\, t_F + q_F^*\, q_F\, \overline{t_F})\, (p_P^* - q_P^*) \right.$$
$$\left. +\ (p_F\, t_F + q_F\, \overline{t_F})\, (a\, p_P^* + \overline{a}\, q_P^*) \right]$$

$$P_4(\mathrm{E_P}, \mathrm{E_F}) = \sum_{T_P^*, T_F^*, B, T_P, T_F} P_4(\mathrm{E_F} \mid T_F) P_4(T_F) P_4(\mathrm{E_P} \mid T_P) P_4(T_P \mid T_P^*)$$
$$\cdot\, P_4(T_P^* \mid T_F^*, B) P_4(B) P_4(T_F^* \mid T_F)$$
$$= p_P \left[\overline{a}\, b\, (p_F^*\, p_F\, t_F + q_F^*\, q_F\, \overline{t_F})\, (p_P^* - q_P^*) \right.$$
$$\left. +\ (p_F\, t_F + q_F\, \overline{t_F})\, (a\, p_P^* + \overline{a}\, q_P^*) \right]$$
$$+\ q_P \left[\overline{a}\, b\, (p_F^*\, p_F\, t_F + q_F^*\, q_F\, \overline{t_F})\, (\overline{p_P^*} - \overline{q_P^*}) \right.$$
$$\left. +\ (p_F\, t_F + q_F\, \overline{t_F})\, (a\, \overline{p_P^*} + \overline{a}\, \overline{q_P^*}) \right]$$

$$P_4(\mathrm{T_P} \mid \mathrm{E_P}) = \frac{P_4(\mathrm{T_P}, \mathrm{E_P})}{P_4(\mathrm{E_P})}$$

$$P_4(\mathrm{T_P}, \mathrm{E_P}) = P_4(\mathrm{E_P} \mid \mathrm{T_P}) \sum_{T_P^*, T_F^*, B, T_F} P_4(\mathrm{T_P} \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B)$$
$$\cdot\, P_4(T_F^* \mid T_F) P_4(T_F)$$
$$= p_P \left[\overline{a}\, b\, (p_F^*\, t_F + q_F^*\, \overline{t_F})\, (p_P^* - q_P^*) + a\, p_P^* + \overline{a}\, q_P^* \right]$$

$$P_4(\mathrm{E_P}) = p_P \left[\overline{a}\, b\, (p_F^*\, t_F + q_F^*\, \overline{t_F})\, (p_P^* - q_P^*) + a\, p_P^* + \overline{a}\, q_P^*\right] + q_P \left[\overline{a}\, b\, (p_F^*\, t_F + q_F^*\, \overline{t_F})\, (\overline{p_P^*} - \overline{q_P^*}) + a\, \overline{p_P^*} + \overline{a}\, \overline{q_P^*}\right]$$

(alternative form of $P_4(\mathrm{E_P})$ from the proof of **Theorem 5**)

$$P_4(\mathrm{T_P} \mid \mathrm{E_P}, \mathrm{E_F}) - P_4(\mathrm{T_P} \mid \mathrm{E_P}) = \frac{\overline{a}\, b\, p_P\, q_P\, t_F\, \overline{t_F}\, (p_F - q_F)\, (p_F^* - q_F^*)\, (p_P^* - q_P^*)}{P_4(\mathrm{E_P}, \mathrm{E_F})\, P_4(\mathrm{E_P})}$$

□

Theorem 8 E_P adds to E_F's confirmation of T_F iff $(p_P - q_P)\, (p_F^* - q_F^*)\, (p_P^* - q_P^*) > 0$.

Proof

$$P_4(\mathrm{T_F} \mid \mathrm{E_F}, \mathrm{E_P}) = \frac{P_4(\mathrm{T_F}, \mathrm{E_F}, \mathrm{E_F})}{P_4(\mathrm{E_F}, \mathrm{E_P})}$$

$$\begin{aligned}
P_4(\mathrm{T_F}, \mathrm{E_F}, \mathrm{E_P}) &= P_4(\mathrm{T_F}) P_4(\mathrm{E_F} \mid \mathrm{T_F}) \sum_{T_P^*, T_F^*, B, T_P} P_4(\mathrm{E_P} \mid T_P) P_4(T_P \mid T_P^*) \\
&\quad \cdot P_4(T_P^* \mid T_F^*, B) P_4(B) P_4(T_F^* \mid \mathrm{T_F}) \\
&= p_F\, t_F \Big[p_F^* \Big(b\, (p_P\, p_P^* + q_P\, \overline{p_P^*}) \\
&\quad + \overline{b}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big) \\
&\quad + \overline{p_F^*}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big]
\end{aligned}$$

$$\begin{aligned}
P_4(\mathrm{E_F}, \mathrm{E_P}) &= p_F\, t_F \Big[p_F^* \Big(b\, (p_P\, p_P^* + q_P\, \overline{p_P^*}) \\
&\quad + \overline{b}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big) \\
&\quad + \overline{p_F^*}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big] \\
&\quad + q_F\, \overline{t_F} \Big[q_F^* \Big(b\, (p_P\, p_P^* + q_P\, \overline{p_P^*}) \\
&\quad + \overline{b}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big) \\
&\quad + \overline{q_F^*}\, (a\, p_P\, p_P^* + a\, q_P\, \overline{p_P^*} + \overline{a}\, p_P\, q_P^* + \overline{a}\, q_P\, \overline{q_P^*})\Big]
\end{aligned}$$

(alternative form of $P_4(\mathrm{E_F}, \mathrm{E_P})$ from the proof of **Theorem 7**)

$$\begin{aligned}
P_4(\mathrm{T_F} \mid \mathrm{E_F}) &= \frac{P_4(\mathrm{T_F}, \mathrm{E_F})}{P_4(\mathrm{E_F})} \\
&= \frac{P_4(\mathrm{E_F} \mid \mathrm{T_F}) P_4(\mathrm{T_F})}{\sum_{T_F} P_4(\mathrm{E_F} \mid T_F) P_4(T_F)} \\
&= \frac{p_F\, t_F}{p_F\, t_F + q_F\, \overline{t_F}}
\end{aligned}$$

$$P_4(\mathrm{T_F} \mid \mathrm{E_F}, \mathrm{E_P}) - P_4(\mathrm{T_F} \mid \mathrm{E_F}) = \frac{\overline{a}\, b\, p_F\, q_F\, t_F\, \overline{t_F}\, (p_P - q_P)\, (p_F^* - q_F^*)\, (p_P^* - q_P^*)}{P_4(\mathrm{E_F}, \mathrm{E_P})\, P_4(\mathrm{E_F})}$$

□

Theorem 7' E_{F} adds to E_{P}'s confirmation of T_{P} iff $(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$\begin{aligned}
P_2(\mathrm{T_P} \mid \mathrm{E_P}, \mathrm{E_F}) &= \frac{P_2(\mathrm{T_P}, \mathrm{E_P}, \mathrm{E_F})}{P_2(\mathrm{E_P}, \mathrm{E_F})} \\
P_2(\mathrm{T_P}, \mathrm{E_P}, \mathrm{E_F}) &= P_2(\mathrm{E_P} \mid \mathrm{T_P}) \sum_{T_P^*, T_F^*, T_F} P_2(\mathrm{T_P} \mid T_P^*) P_2(T_P^* \mid T_F^*) \\
&\quad \cdot P_2(T_F^* \mid T_F) P_2(\mathrm{E_F} \mid T_F) P_2(T_F) \\
&= p_P \left[p_F\, t_F\, (p_P^*\, p_F^* + q_P^*\, \overline{p_F^*}) + q_F\, \overline{t_F}\, (p_P^*\, q_F^* + q_P^*\, \overline{q_F^*}) \right] \\
P_2(\mathrm{E_P}, \mathrm{E_F}) &= \sum_{T_P^*, T_F^*, T_P, T_F} P_2(\mathrm{E_F} \mid T_F) P_2(T_F) P_2(\mathrm{E_P} \mid T_P) P_2(T_P \mid T_P^*) \\
&\quad \cdot P_2(T_P^* \mid T_F^*) P_2(T_F^* \mid T_F) \\
&= p_P \left[p_F\, t_F\, (p_P^*\, p_F^* + q_P^*\, \overline{p_F^*}) + q_F\, \overline{t_F}\, (p_P^*\, q_F^* + q_P^*\, \overline{q_F^*}) \right] \\
&\quad + q_P \left[p_F\, t_F\, (\overline{p_P^*}\, p_F^* + \overline{q_P^*}\, \overline{p_F^*}) + q_F\, \overline{t_F}\, (\overline{p_P^*}\, q_F^* + \overline{q_P^*}\, \overline{q_F^*}) \right] \\
P_2(\mathrm{T_P} \mid \mathrm{E_P}) &= \frac{P_2(\mathrm{T_P}, \mathrm{E_P})}{P_2(\mathrm{E_P})} \\
P_2(\mathrm{T_P}, \mathrm{E_P}) &= P_2(\mathrm{E_P} \mid \mathrm{T_P}) \sum_{T_P^*, T_F^*, T_F} P_2(\mathrm{T_P} \mid T_P^*) P_2(T_P^* \mid T_F^*) \\
&\quad \cdot P_2(T_F^* \mid T_F) P_2(T_F) \\
&= p_P \left[t_F\, (p_P^*\, p_F^* + q_P^*\, \overline{p_F^*}) + \overline{t_F}\, (p_P^*\, q_F^* + q_P^*\, \overline{q_F^*}) \right] \\
P_2(\mathrm{E_P}) &= \sum_{T_P^*, T_F^*, T_P, T_F} P_2(\mathrm{E_P} \mid T_P) P_2(T_P \mid T_P^*) P_2(T_P^* \mid T_F^*) \\
&\quad \cdot P_2(T_F^* \mid T_F) P_2(T_F) \\
&= p_P \left[t_F\, (p_P^*\, p_F^* + q_P^*\, \overline{p_F^*}) + \overline{t_F}\, (p_P^*\, q_F^* + q_P^*\, \overline{q_F^*}) \right] \\
&\quad \cdot q_P \left[t_F\, (\overline{p_P^*}\, p_F^* + \overline{q_P^*}\, \overline{p_F^*}) + \overline{t_F}\, (\overline{p_P^*}\, q_F^* + \overline{q_P^*}\, \overline{q_F^*}) \right] \\
P_2(\mathrm{T_P} \mid \mathrm{E_P}, \mathrm{E_F}) - P_2(\mathrm{T_P} \mid \mathrm{E_P}) &= \frac{p_P\, q_P\, t_F\, \overline{t_F}\, (p_F - q_F)\, (p_F^* - q_F^*)\, (p_P^* - q_P^*)}{P_2(\mathrm{E_P}, \mathrm{E_F})\, P_2(\mathrm{E_P})}
\end{aligned}$$

□

Theorem 8' E_{P} adds to E_{F}'s confirmation of T_{F} iff $(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*) > 0$.

Proof

$$P_2(\mathrm{T_F} \mid \mathrm{E_F}, \mathrm{E_P}) = \frac{P_2(\mathrm{T_F}, \mathrm{E_F}, \mathrm{E_P})}{P_2(\mathrm{E_F}, \mathrm{E_P})}$$

$$\begin{aligned} P_2(\mathrm{T_F}, \mathrm{E_F}, \mathrm{E_P}) &= P_2(\mathrm{T_F}) P_2(\mathrm{E_F} \mid \mathrm{T_F}) \sum_{T_P^*, T_F^*, T_P} P_2(T_P \mid T_P^*) P_2(T_P^* \mid T_F^*) \\ &\quad \cdot P_2(T_F^* \mid \mathrm{T_F}) P_2(\mathrm{E_P} \mid T_P) \\ &= p_F\, t_F \left[p_F^* \,(p_P\, p_P^* + q_P\, \overline{p_P^*}) + \overline{p_F^*}\,(p_P\, q_P^* + q_P\, \overline{q_P^*}) \right] \\ P_2(\mathrm{E_F}, \mathrm{E_P}) &= p_F\, t_F \left[p_F^* \,(p_P\, p_P^* + q_P\, \overline{p_P^*}) + \overline{p_F^*}\,(p_P\, q_P^* + q_P\, \overline{q_P^*}) \right] \\ &\quad \cdot q_F\, \overline{t_F} \left[q_F^* \,(p_P\, p_P^* + q_P\, \overline{p_P^*}) + \overline{q_F^*}\,(p_P\, q_P^* + q_P\, \overline{q_P^*}) \right] \end{aligned}$$

(alternative form of $P_2(\mathrm{E_F}, \mathrm{E_P})$ from the proof of **Theorem 7'**)

$$\begin{aligned} P_2(\mathrm{T_F} \mid \mathrm{E_F}) &= \frac{P_2(\mathrm{T_F}, \mathrm{E_F})}{P_2(\mathrm{E_F})} \\ &= \frac{P_2(\mathrm{E_F} \mid \mathrm{T_F}) P_2(\mathrm{T_F})}{\sum_{T_F} P_2(\mathrm{E_F} \mid T_F) P_2(T_F)} \\ &= \frac{p_F\, t_F}{p_F\, t_F + q_F\, \overline{t_F}} \end{aligned}$$

$$P_2(\mathrm{T_F} \mid \mathrm{E_F}, \mathrm{E_P}) - P_2(\mathrm{T_F} \mid \mathrm{E_F}) = \frac{p_F\, q_F\, t_F\, \overline{t_F}\,(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{P_2(\mathrm{E_F}, \mathrm{E_P})\, P_2(\mathrm{E_F})}$$

□

Theorem 9 Given a, p_F, q_F, p_F^*, q_F^*, p_P^*, q_P^*, and t_F are constant and $p_F > q_F$, $p_F^* > q_F^*$, and $p_P^* > q_P^*$, if b increases (decreases), then $d(T_\mathrm{P}, E_\mathrm{F})$ increases (decreases).

Proof From the proof of the Theorem 5 above, we have that:

$$P_4(\mathrm{T_P} \mid \mathrm{E_F}) - P_4(\mathrm{T_P}) = \frac{\overline{a}\, b\, t_F\, \overline{t_F}\,(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{p_F\, t_F + q_F\, \overline{t_F}}$$

Observe that, given t_F, a, p_F, q_F, p_F^*, q_F^*, p_P^*, and q_P^* are constant and $p_F > q_F$, $p_F^* > q_F^*$, and $p_P^* > q_P^*$, if b increases, then $\overline{a}\, b\, t_F\, \overline{t_F}\,(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)$ increases. As the denominator, i.e. $p_F\, t_F + q_F\, \overline{t_F}$, does not dependent on b, then if b increases, $\frac{\overline{a}\, b\, t_F\, \overline{t_F}\,(p_F - q_F)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{p_F\, t_F + q_F\, \overline{t_F}}$ increases, i.e. $d(\mathrm{T_P}, \mathrm{E_F})$ increases.

□

Theorem 10 Given a, p_P, q_P, p_F^*, q_F^*, p_P^*, q_P^*, and t_F are constant and $p_P > q_P$, $p_F^* > q_F^*$, and $p_P^* > q_P^*$, if b increases (decreases), then $d(T_\mathrm{F}, E_\mathrm{P})$ increases (decreases).

Proof From the proof of the Theorem 6 above, we have that:

$$P_4(\mathrm{T_F} \mid \mathrm{E_P}) - P_4(\mathrm{T_F}) = \frac{\overline{a}\, b\, t_F\, \overline{t_F}\,(p_P - q_P)\,(p_F^* - q_F^*)\,(p_P^* - q_P^*)}{P_4(\mathrm{E_P})}$$

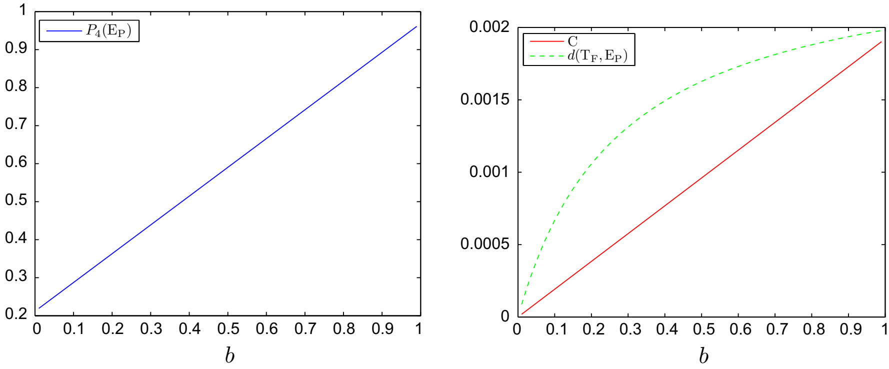

Fig. 7 Dependence of $P_4(\mathrm{E_P})$ (*left figure*) and C and $d(\mathrm{T_F}, \mathrm{E_P})$ (*right figure*) on b

Notice that, in contrast to the previous proof, the denominator, i.e. $P_4(\mathrm{E_P})$, is dependent on b; so, changing the value of b would also change the value of $P_4(\mathrm{E_P})$. Alternative way of writing $P_4(\mathrm{E_P})$ so that it better serves the purpose of this proof is:

$$P_4(\mathrm{E_P}) = \overline{a}\, b\, (p^*_F\, t_F + q^*_F\, \overline{t_F})\, (p_P - q_P)\, (p^*_P - q^*_P) + a\, (p_P\, p^*_P + q_P\, \overline{p^*_P}) \\ + \overline{a}\, (p_P\, q^*_P + q_P\, \overline{q^*_P})$$

Further, let us introduce the following abbreviations:

$$\mathrm{C} := \overline{a}\, b\, t_F\, \overline{t_F}\, (p_P - q_P)\, (p^*_F - q^*_F)\, (p^*_P - q^*_P)$$
$$\mathrm{D} := \overline{a}\, b\, (p^*_F\, t_F + q^*_F\, \overline{t_F})\, (p_P - q_P)\, (p^*_P - q^*_P)$$

Observe that both C and $P_4(\mathrm{E_P})$ increase as b increases (other values remaining constant). To see which of the two, C or $P_4(\mathrm{E_P})$, increases faster with the increase of b, we calculate C − D.

$$\mathrm{C} - \mathrm{D} = -\overline{a}\, b\, (p_P - q_P)\, (p^*_P - q^*_P)\, (t^2_F\, p^*_F + t_F\, \overline{t_F}\, q^*_F + \overline{t_F}\, q^*_F)$$

So, given $p_P > q_P$ and $p^*_P > q^*_P$, $\mathrm{C} < \mathrm{D}$ (as a consequence $\mathrm{C} < P_4(\mathrm{E_P})$; so $\frac{\mathrm{C}}{P_4(\mathrm{E_P})} < 1$). Hence, with the increase of b, $P_4(\mathrm{E_P})$ increases faster than C, that is, the slope of $P_4(\mathrm{E_P})$ is greater than the slope of C. Nevertheless, even with a very large slope of $P_4(\mathrm{E_P})$ and a very small slope of C, $\frac{\mathrm{C}}{P_4(\mathrm{E_P})}$ still increases, as shown in Fig. 7. So, if b increases, $d(\mathrm{T_F}, \mathrm{E_P})$ increases. □

Theorem 11 $\Delta_0 = 0$ iff $(p^*_F = q^*_F)$ or $(p^*_P = q^*_P)$. And $\Delta_0 > 0$ if $(p^*_F > q^*_F)$ and if $(p^*_P > q^*_P)$.

Proof

$$
\begin{aligned}
P_1(\mathrm{T_F}, \mathrm{T_P}) &= t_F\, t_P \\
&= t_F \left[p_P^* \left(b\, p_F^*\, t_F + b\, q_F^*\, \overline{t_F} + a\, \overline{b} + a\, b\, \overline{p_F^*}\, t_F + a\, b\, \overline{q_F^*}\, \overline{t_F} \right) \right. \\
&\quad \left. + \; q_P^* \left(\overline{a}\, \overline{b} + \overline{a}\, b\, \overline{p_F^*}\, t_F + \overline{a}\, b\, \overline{q_F^*}\, \overline{t_F} \right) \right] (P_4(\mathrm{T_P}) \text{ instead of } P_1(\mathrm{T_P}) \\
P_4(\mathrm{T_F}, \mathrm{T_P}) &= P_4(\mathrm{T_F}) \sum_{T_P^*, T_F^*, B} P_4(\mathrm{T_P} \mid T_P^*) P_4(T_P^* \mid T_F^*, B) P_4(B) P_4(T_F^* \mid \mathrm{T_F}) \\
&= t_F \left[p_P^* \, (b\, p_F^* + a\, \overline{b} + a\, b\, \overline{p_F^*}) + q_P^* \, (\overline{a}\, \overline{b} + \overline{a}\, b\, \overline{p_F^*}) \right] \\
\Delta_0 &:= P_4(\mathrm{T_F}, \mathrm{T_P}) - P_1(\mathrm{T_F}, \mathrm{T_P}) \\
\Delta_0 &= \overline{a}\, b\, t_F\, \overline{t_F}\, (p_F^* - q_F^*)\, (p_P^* - q_P^*)
\end{aligned}
$$

□

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